Particular Cases of Special Robertson-Walker Multiply Warped Products Having an Affine Connection[1]

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Abstract

In this paper we study some particular cases of Einstein equations for special Robertson-Walker multiply warped products $M = I \times_{f_1} F_1 \times ... \times_{f_m} F_m$*, where* $I \subset \mathbb{R}$ *is an open interval,* $\dim I = 1$, $f_i : I \longrightarrow (0, \infty)$, $f_i \in \mathcal{C}^{\infty}(I)$, $\dim F_i = k_i \geq 1$ *for every* $i \in \{1, ..., m\}$, $m \geq 1$, *having an affine connection. We compute the warping functions in the following cases:*

A). M is Ricci flat having a quarter-symmetric non-metric connection and all the fibres have the dimensions equal to 1.

B). M is Ricci flat having a quarter-symmetric metric connection and all the fibres have the dimensions equal to 1.

C). M is Ricci flat having a quarter-symmetric metric connection and all the the warping functions are equal.

D). M is Ricci flat having a quarter-symmetric non-metric connection and all the warping functions are equal.

E). M is Ricci flat having a semi-symmetric non-metric connection with the supplementary condition $H = \sum\limits_{i=1}^{m} k_i \frac{f_i^{'}}{f_i} = 0$.

F). M is Ricci flat with a quarter-symmetric non-metric connection with the supplementary condition $H = \sum\limits_{i=1}^{m} k_i \frac{f_i^{'}}{f_i} = (\overline{n} - 1)\lambda_1 - \lambda_2$.

Keywords: Einstein space, multiply warped product, warping function, quarter-symmetric connection.

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1. *Introduction*

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection on M.

Definition 1.1. ([7], [8], [9], [10]) a). A linear connection $\overline{\nabla}$ on M is called a *quarter-symmetric connection* if the torsion tensor T of $\overline{\nabla}$, $T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y]$, satisfies $T(X, Y) = \pi(Y)\phi X - \pi(X)\phi Y$, where π is a

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1-form associated with the vector field P on M defined by $\pi(X) = g(X, P)$ and ϕ is a $(1,1)$ tensor field.

b). $\overline{\nabla}$ is called a *quarter-symmetric metric connection* if $\overline{\nabla} g = 0$.

c). $\overline{\nabla}$ is called a *quarter-symmetric non-metric connection* if it satisfies $\overline{\nabla} g \neq 0$.

d). A linear connection $\overline{\nabla}$ on M is called a *semi-symmetric connection* if the torsion tensor T of $\overline{\nabla}$, $T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]$, satisfies $T(X,Y) = \pi(Y)X - \pi(X)Y$, where π is a 1-form associated with the vector field P on M defined by $\pi(X) = g(X,P)$.

e). $\overline{\nabla}$ is called a *semi-symmetric metric connection* if $\overline{\nabla} g = 0$.

f). $\overline{\nabla}$ is called a *semi-symmetric non-metric connection* if it satisfies $\overline{\nabla} g \neq 0$.

We have the following two well-known results:

Theorem 1.1. *(theorem 3.1, p. 7, [9], semi-symmetric non-metric connection) Let* $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times ... \times_{f_m} F_m$ *be a multiply warped product with the metric tensor* $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus ... \oplus f_m^2 g_{F_m}$ *and* $P = \left(\frac{\partial}{\partial t}\right)$. *Then* $(M, \overline{\nabla})$ *is Einstein with the Einstein constant* λ *if and only if the following conditions are satisfied for every* $i \in \{1, ..., m\}$

$$\begin{cases} \left(F_i, \nabla^{F_i}\right) \text{ is Einstein with Einstein constant } \lambda_i \\ \sum\limits_{i=1}^{m} k_i \left(1 - \frac{f_i''}{f_i}\right) = \lambda \\ \lambda_i - f_i f_i'' - (k_i - 1)\left(f_i'\right)^2 - f_i f_i' \sum\limits_{j \neq i,\, j=1}^{m} k_j \frac{f_j'}{f_j} + f_i^2 \sum\limits_{j=1}^{m} k_j \frac{f_j'}{f_j} = \lambda f_i^2 \end{cases} \tag{1}$$

In the sequence we will consider special multiply Einstein warped products. Thus let $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times ... \times_{f_m} F_m$, $m \geq 1$, $\dim M = \overline{n} \geq 3$, $\dim I = 1$ ($I \subset \mathbb{R}$ is an open interval), $\dim F_i = k_i \geq 1$, $f_i : I \longrightarrow (0, \infty)$, $f_i \in \mathcal{C}^\infty(I)$ for every $i \in \{1, ..., m\}$ and $\overline{n} = 1 + \sum\limits_{i=1}^{m} k_i \geq 4$.

Theorem 1.2. *(theorem 4.11, p. 23, [10], quarter-symmetric non-metric connection) Let* $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times ... \times_{f_m} F_m$ *be a multiply warped product with the metric tensor* $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus ... \oplus f_m^2 g_{F_m}$ *and* $P = \left(\frac{\partial}{\partial t}\right)$. *Then* $(M, \overline{\nabla})$ *is Einstein with the Einstein constant* α *if and only if the following conditions are satisfied for any* $i \in \{1, ..., m\}$*:*

$$\begin{cases} \left(F_i, \nabla^{F_i}\right) \text{ is Einstein with Einstein constant } \alpha_i \\ \sum\limits_{i=1}^{m} k_i \left(\lambda_2 \frac{f_i'}{f_i} - \frac{f_i''}{f_i} + \lambda_1^2 - \lambda_1 \lambda_2\right) = \alpha \\ \alpha_i - f_i f_i'' - (k_i - 1)\left(f_i'\right)^2 + \left(\lambda_2 f_i^2 - f_i f_i'\right) \sum\limits_{j \neq i,\, j=1}^{m} k_j \frac{f_j'}{f_j} + \\ \quad + \left[\lambda_2^2 + (1 - \overline{n})\lambda_1 \lambda_2\right] f_i^2 + \left[(\overline{n} - 1)\lambda_1 + (k_i - 1)\lambda_2\right] f_i f_i' = \alpha f_i^2 \end{cases} \tag{2}$$

Remark 1.1. ([10]) The scalars $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0,1\}$. If $\lambda_1 \neq \lambda_2$ then $\overline{\nabla}$ is a quarter-symmetric non-metric connection, if $\lambda_1 = \lambda_2$ then $\overline{\nabla}$ is a quarter-symmetric metric connection.

Hence the Einstein equations for special Robertson-Walker multiply warped products having a quarter-symmetric metric connection can be deduced from (2) and have the following form for every $i \in \{1, ..., m\}$ where $\lambda_1 = \lambda_2 = \lambda \notin \{0,1\}$:

$$\begin{cases} \left(F_i, \nabla^{F_i}\right) \text{ is Einstein with Einstein constant } \alpha_i \\ \sum\limits_{i=1}^{m} k_i \left(\lambda \frac{f_i^{'}}{f_i} - \frac{f_i^{''}}{f_i}\right) = \alpha \\ \alpha_i - f_i f_i^{''} - (k_i - 1)\left(f_i^{'}\right)^2 + \left(\lambda f_i^2 - f_i f_i^{'}\right) \sum\limits_{j \neq i,\ j=1}^{m} k_j \frac{f_j^{'}}{f_j} + \\ \qquad + \left[\lambda^2 + (1-\overline{n})\lambda^2\right] f_i^2 + \left[(\overline{n}-1)\lambda + (k_i - 1)\lambda\right] f_i f_i^{'} = \alpha f_i^2 \end{cases} \tag{3}$$

2. ***Main results***

The aim of this article is to solve the systems (2) and (3) in the particular case, when M is Ricci flat, that is $\alpha = 0$ and all the fibres have the dimensions equal to one, that is $k_i = 1$ for every $i \in \{1, ..., m\}$; this implies $\alpha_i = 0$ for every $i \in \{1, ..., m\}$, or when all the warping functions are equal. We also consider $m \geq 3$ and remark that $\overline{n} = m + 1$. Moreover, we consider the cases of equal warping functions and the cases of the systems (1) and (2) with supplementary conditions.

A). M is Ricci flat having a quarter-symmetric non-metric connection and $\dim F_i = 1$ for every $i \in \{1, ..., m\}$. Hence the system (2) becomes for every $i \in \{1, ..., m\}$:

$$\begin{cases} \sum\limits_{i=1}^{m} \left(\lambda_2 \frac{f_i^{'}}{f_i} - \frac{f_i^{''}}{f_i} + \lambda_1^2 - \lambda_1\lambda_2\right) = 0 \\ -f_i f_i^{''} + \left(\lambda_2 f_i^2 - f_i f_i^{'}\right) \sum\limits_{j \neq i,\ j=1}^{m} \frac{f_j^{'}}{f_j} + (\lambda_2^2 - m\lambda_1\lambda_2) f_i^2 + m\lambda_1 f_i f_i^{'} = 0 \end{cases} \tag{4}$$

Dividing by f_i^2 the second equation of (4) one can obtain the equivalent form:

$$\begin{cases} \sum\limits_{i=1}^{m} \left(\lambda_2 \frac{f_i^{'}}{f_i} - \frac{f_i^{''}}{f_i} + \lambda_1^2 - \lambda_1\lambda_2\right) = 0 \\ -\frac{f_i^{''}}{f_i} + \left(\lambda_2 - \frac{f_i^{'}}{f_i}\right) \sum\limits_{j \neq i,\ j=1}^{m} \frac{f_j^{'}}{f_j} + (\lambda_2^2 - m\lambda_1\lambda_2) + m\lambda_1 \frac{f_i^{'}}{f_i} = 0 \\ \qquad \text{for every } i \in \{1, ..., m\} \end{cases} \tag{5}$$

We make now the following notations: $h_i = \frac{f_i^{'}}{f_i}$ for every $i \in \{1, ..., m\}$ and $H = \sum\limits_{i=1}^{m} \frac{f_i^{'}}{f_i} = \sum\limits_{i=1}^{m} h_i$. We remark that $\frac{f_i^{''}}{f_i} = h_i^2 + h_i^{'}$ for every $i \in \{1, ..., m\}$. The system (5) becomes

$$\begin{cases} \sum\limits_{i=1}^{m} \left[\lambda_2 h_i - h_i^{'} - h_i^2 + \lambda_1^2 - \lambda_1 \lambda_2\right] = 0 \\ h_i^{'} = (\lambda_2 - h_i)(H + \lambda_2 - m\lambda_1) \text{ for every } i \in \{1, ..., m\} \end{cases} \tag{6}$$

Summing over i the above m equations of (6) we obtain

$$H^{'} = (H + \lambda_2 - m\lambda_1)(m\lambda_2 - H).$$

We also denote by $A = (m+1)\lambda_2 - m\lambda_1$. Thus:

I). If $A = 0$, then:

I.1). If $H = m\lambda_1 - \lambda_2$, then $h_i^{'} = 0 \Longrightarrow h_i(x) = \varepsilon_i$ for every $i \in \{1, ..., m\}$. From $\frac{f_i^{'}}{f_i} = h_i = \varepsilon_i$ we obtain $f_i(x) = \eta_i e^{\varepsilon_i x}$, with $\eta_i > 0$ for every $i \in \{1, ..., m\}$. Since $H = \sum\limits_{i=1}^{m} h_i$ we obtain that $\sum\limits_{i=1}^{m} \varepsilon_i = m\lambda_1 - \lambda_2$. The first relation implies:

$$\sum_{i=1}^{m} \left[\lambda_2 h_i - h_i^{'} - h_i^2 + \lambda_1^2 - \lambda_1\lambda_2\right] = 0 \Longrightarrow$$
$$\sum_{i=1}^{m} \varepsilon_i^2 = \lambda_2 \sum_{i=1}^{m} \varepsilon_i + m\left(\lambda_1^2 - \lambda_1\lambda_2\right) \Longrightarrow$$
$$\sum_{i=1}^{m} \varepsilon_i^2 = m\lambda_1^2 - \lambda_2^2$$

Making the change of variable $\varepsilon_i = \lambda_2 - c_i$ for every $i \in \{1, ..., m\}$ then we obtain the following solution: $f_i(x) = \eta_i e^{(\lambda_2 - c_i)x}$, with $\eta_i > 0$, $c_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$ such that

$$\sum_{i=1}^{m} c_i = (m+1)\lambda_2 - m\lambda_1 = A = 0$$

and

$$\sum_{i=1}^{m} c_i^2 = \sum_{i=1}^{m} \left[\lambda_2^2 - 2\lambda_2\varepsilon_i + \varepsilon_i^2\right] = (m+1)\lambda_2^2 - 2m\lambda_1\lambda_2 + m\lambda_1^2.$$

I.2). If $H = m\lambda_2$, then $h_i^{'} = (\lambda_2 - h_i)(m\lambda_2 + \lambda_2 - m\lambda_1) = (\lambda_2 - h_i) A = 0$. This implies $h_i(x) = \varepsilon_i$, with $\varepsilon_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$. From $\frac{f_i^{'}}{f_i} = h_i = \varepsilon_i$

we obtain $f_i(x) = \eta_i e^{\varepsilon_i x}$, with $\eta_i > 0$ for every $i \in \{1, ..., m\}$. Since $H = \sum_{i=1}^{m} h_i$ we obtain that $\sum_{i=1}^{m} \varepsilon_i = m\lambda_2$. The first relation implies:

$$\sum_{i=1}^{m} \left[\lambda_2 h_i - h_i^{'} - h_i^{2} + \lambda_1^2 - \lambda_1\lambda_2\right] = 0 \Longrightarrow$$
$$\sum_{i=1}^{m} \varepsilon_i^2 = \lambda_2 \sum_{i=1}^{m} \varepsilon_i + m\left(\lambda_1^2 - \lambda_1\lambda_2\right) \Longrightarrow$$
$$\sum_{i=1}^{m} \varepsilon_i^2 = m\lambda_2^2 - m\lambda_1\lambda_2 + m\lambda_1^2$$

Making the change of variable $\varepsilon_i = \lambda_2 - c_i$ for every $i \in \{1, ..., m\}$ then we obtain the following solution: $f_i(x) = \eta_i e^{(\lambda_2 - c_i)x}$, with $\eta_i > 0$, $c_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$ such that

$$\sum_{i=1}^{m} c_i = m\lambda_2 - \sum_{i=1}^{m} \varepsilon_i = m\lambda_2 - m\lambda_2 = 0 = A$$

and

$$\sum_{i=1}^{m} c_i^2 = -m\lambda_1\lambda_2 + m\lambda_1^2 =$$
$$(m+1)\lambda_2^2 - 2m\lambda_1\lambda_2 + m\lambda_1^2 - (m+1)\lambda_2^2 + m\lambda_1\lambda_2 =$$
$$(m+1)\lambda_2^2 - 2m\lambda_1\lambda_2 + m\lambda_1^2 - \lambda_2 A = (m+1)\lambda_2^2 - 2m\lambda_1\lambda_2 + m\lambda_1^2.$$

I.3). If $H \neq m\lambda_1 - \lambda_2$ (and $H \neq m\lambda_2 = m\lambda_1 - \lambda_2$), then

$$H^{'} = -H^2 + H\left(m\lambda_2 - \lambda_2 + m\lambda_1\right) + \lambda_2\left(m\lambda_2 - m^2\lambda_1\right) =$$
$$-\left(H^2 - 2m\lambda_2 H + m^2\lambda_2^2\right) = -(H - m\lambda_2)^2.$$

This implies $H(x) = m\lambda_2 + \frac{1}{x+c}$, where $c \in \mathbb{R}$. Since

$$\begin{aligned} h_i^{'} = \; & (\lambda_2 - h_i)\left(\tfrac{1}{x+c} + m\lambda_2 + \lambda_2 - m\lambda_1\right) = \\ & (\lambda_2 - h_i)\left(\tfrac{1}{x+c} + A\right) = \\ & (\lambda_2 - h_i)\tfrac{1}{x+c}, \end{aligned}$$

we can obtain the solution (chosing the positive modulus, the case of negative modulus being analogue) of the following form: $h_i(x) = \frac{\lambda_2 x + c_i}{x+c}$, with $c_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$. We remark that since $H = \sum_{i=1}^{m} h_i$ we obtain:

$$\frac{1}{x+c} + m\lambda_2 = \frac{m\lambda_2 x + \sum\limits_{i=1}^{m} c_i}{x+c} \Longrightarrow$$
$$1 + m\lambda_2(x+c) = m\lambda_2 x + \sum_{i=1}^{m} c_i \Longrightarrow$$
$$\sum_{i=1}^{m} c_i = m\lambda_2 c + 1$$

We also have $h_i^{'}(x) = \frac{\lambda_2 c - c_i}{(x+c)^2}$ and $h_i^2(x) = \frac{\lambda_2^2 x^2 + 2\lambda_2 c_i x + c_i^2}{(x+c)^2}$. Therefore, the first relation becomes:

$$\sum_{i=1}^{m} \left[\lambda_2 h_i - h_i^{'} - h_i^{2} + \lambda_1^2 - \lambda_1\lambda_2\right] = 0 \Longrightarrow$$
$$\sum_{i=1}^{m} \left(\lambda_2^2 cx + \lambda_2 c c_i - \lambda_2 c + c_i - \lambda_2 c_i x - c_i^2\right) =$$
$$-m\left(\lambda_1^2 - \lambda_1\lambda_2\right)\left(x^2 + 2xc + c^2\right) \Longrightarrow \tag{7}$$
$$\Longrightarrow \begin{cases} -m\lambda_1\left(\lambda_1 - \lambda_2\right) = 0 \Longrightarrow \lambda_1 = \lambda_2 \text{ (contradiction with } \lambda_1 \neq \lambda_2) \\ -\lambda_2 = -2cm\left(\lambda_1^2 - \lambda_1\lambda_2\right) \\ m\lambda_2^2 c^2 + \lambda_2 c + 1 - \sum\limits_{i=1}^{m} c_i^2 = -m\left(\lambda_1^2 - \lambda_1\lambda_2\right)c^2 \end{cases}$$

Hence we don't have any solution in this case.

II). If $A \neq 0$, then:

II.1). If $H = m\lambda_1 - \lambda_2$, then $h_i^{'} = (\lambda_2 - h_i)(m\lambda_1 - \lambda_2 + \lambda_2 - m\lambda_1) = 0 \Longrightarrow$ $h_i(x) = \varepsilon_i$ for every $i \in \{1, ..., m\}$. From $\frac{f_i^{'}}{f_i} = h_i = \varepsilon_i$ we obtain $f_i(x) = \eta_i e^{\varepsilon_i x}$, with $\eta_i > 0$ for every $i \in \{1, ..., m\}$. Since $H = \sum\limits_{i=1}^{m} h_i$ we obtain that $\sum\limits_{i=1}^{m} \varepsilon_i = m\lambda_1 - \lambda_2$. The first relation implies:

$$\sum_{i=1}^{m} \left[\lambda_2 h_i - h_i^{'} - h_i^{2} + \lambda_1^2 - \lambda_1\lambda_2\right] = 0 \Longrightarrow$$
$$\sum_{i=1}^{m} \varepsilon_i^2 = \lambda_2 \sum_{i=1}^{m} \varepsilon_i + m\left(\lambda_1^2 - \lambda_1\lambda_2\right) \Longrightarrow$$
$$\sum_{i=1}^{m} \varepsilon_i^2 = m\lambda_1^2 - \lambda_2^2$$

Making the change of variable $\varepsilon_i = \lambda_2 - c_i$ for every $i \in \{1, ..., m\}$ then we obtain the following solution: $f_i(x) = \eta_i e^{(\lambda_2 - c_i)x}$, with $\eta_i > 0$, $c_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$ such that $\sum_{i=1}^{m} c_i = m\lambda_2 - \sum_{i=1}^{m} \varepsilon_i = (m+1)\lambda_2 - m\lambda_1 = A$ and $\sum_{i=1}^{m} c_i^2 = \sum_{i=1}^{m} \left[\lambda_2^2 - 2\lambda_2\varepsilon_i + \varepsilon_i^2\right] = (m+1)\lambda_2^2 - 2m\lambda_1\lambda_2 + m\lambda_1^2$.

II.2). If $H = m\lambda_2$, then $h_i^{'} = (\lambda_2 - h_i)(m\lambda_2 + \lambda_2 - m\lambda_1) = (\lambda_2 - h_i)A$. This implies $h_i(x) = \lambda_2 + c_i e^{-AX}$, with $c_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$. Since $H = \sum_{i=1}^{m} h_i$ we obtain that:

$$
\begin{gathered}
m\lambda_2 = m\lambda_2 + e^{-Ax}\sum_{i=1}^{m} c_i \Longrightarrow \\
e^{-Ax}\sum_{i=1}^{m} c_i = 0 \Longrightarrow \\
\sum_{i=1}^{m} c_i = 0
\end{gathered}
$$

The first relation implies:

$$
\begin{gathered}
\sum_{i=1}^{m} \left[\lambda_2 h_i - h_i^{'} - h_i^{2} + \lambda_1^2 - \lambda_1\lambda_2\right] = 0 \Longrightarrow \\
Ae^{-Ax}\sum_{i=1}^{m} c_i - \lambda_2 e^{-Ax}\sum_{i=1}^{m} c_i - e^{-2Ax}\sum_{i=1}^{m} c_i^2 = -m\left(\lambda_1^2 - \lambda_1\lambda_2\right) \Longrightarrow \\
e^{-2Ax}\sum_{i=1}^{m} c_i^2 = m\left(\lambda_1^2 - \lambda_1\lambda_2\right) \Longrightarrow \\
\Longrightarrow \begin{cases} \sum_{i=1}^{m} c_i^2 = \\ m\left(\lambda_1^2 - \lambda_1\lambda_2\right) = 0 \Longrightarrow \\ \lambda_1 = \lambda_2 (\text{contradiction with} \lambda_1 \neq \lambda_2) \end{cases}
\end{gathered}
\tag{8}
$$

Hence we don't have any solution in this case.

II.3). If $H \neq m\lambda_1 - \lambda_2$ and $H \neq m\lambda_2$, then

$$
\begin{gathered}
H^{'} = -H^2 + H\left[(m-1)\lambda_2 + m\lambda_1\right] + \lambda_2\left(m\lambda_2 - m^2\lambda_1\right) = \\
-\left[H - \tfrac{(m-1)\lambda_2 + m\lambda_1}{2}\right]^2 + \tfrac{\left[(m+1)\lambda_2 - m\lambda_1\right]^2}{4} = \\
-\left(H - \tfrac{B}{2}\right)^2 + \left(\tfrac{A}{2}\right)^2
\end{gathered}
$$

where $B = (m-1)\lambda_2 + m\lambda_1$. We obtain that $A + B = 2m\lambda_2$ and $B - A = -2\lambda_2 + 2m\lambda_1$. We also denote by $\delta = \frac{B+A}{2} = m\lambda_2$ and $\varepsilon = \frac{B-A}{2} = -\lambda_2 + m\lambda_1$.

Hence, considering the positive modulus, the case of negative modulus being analogue, we obtain the solution of the form $H(x) = \frac{\delta e^{Ax} - \varepsilon K}{e^{Ax} - K} = \delta + \frac{AK}{e^{Ax} - K}$, where $K > 0$. Remark that $\delta - \varepsilon = A$.

Since $h_i^{'} = (\lambda_2 - h_i)\left(\delta + \frac{AK}{e^{Ax}-K} - \varepsilon\right) = (\lambda_2 - h_i)\left(A + \frac{AK}{e^{Ax}-K}\right)$, we can obtain the solution of the following form: $h_i(x) = \frac{\lambda_2 e^{Ax} + c_i}{e^{Ax} - K}$, with $c_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$. We remark that since $H = \sum\limits_{i=1}^{m} h_i$ we obtain:

$$\delta + \frac{AK}{e^{Ax} - K} = \frac{m\lambda_2 e^{Ax} + \sum\limits_{i=1}^{m} c_i}{e^{Ax} - K} \Longrightarrow$$

$$\delta e^{Ax} - \delta K + AK = m\lambda_2 e^{Ax} + \sum_{i=1}^{m} c_i \Longrightarrow$$

$$\begin{cases} m\lambda_2 = \delta \\ -\delta K + AK = \sum\limits_{i=1}^{m} c_i \end{cases} \Longrightarrow \sum_{i=1}^{m} c_i = -\varepsilon K$$

We remark that $A - \delta = -\varepsilon$. We also have

$$h_i^{'}(x) = \frac{\lambda_2 A e^{Ax}}{e^{Ax} - K} - \frac{A\lambda_2 e^{2Ax} + c_i A e^{Ax}}{\left(e^{Ax} - K\right)^2}$$

and

$$h_i^2(x) = \frac{\lambda_2^2 e^{2Ax} + 2\lambda_2 c_i e^{Ax} + c_i^2}{\left(e^{Ax} - K\right)^2}.$$

Thus the first relation becomes:

$$\sum_{i=1}^{m} \left[\lambda_2 h_i - h_i^{'} - h_i^2 + \lambda_1^2 - \lambda_1\lambda_2\right] = 0 \Longrightarrow$$

$$\sum_{i=1}^{m} \left[e^{Ax}\left(-\lambda_2^2 K + \lambda_2 AK + Ac_i - \lambda_2 c_i\right) - \left(\lambda_2 c_i K + c_i^2\right)\right] =$$

$$-m\left(\lambda_1^2 - \lambda_1\lambda_2\right)\left(e^{2Ax} - 2Ke^{Ax} + K^2\right) \Longrightarrow$$

$$e^{Ax}\left(-m\lambda_2^2 K + m\lambda_2 AK - AK\varepsilon + \lambda_2 K\varepsilon\right) + \lambda_2 K^2 \varepsilon - \sum_{i=1}^{m} c_i^2 =$$

$$-m\left(\lambda_1^2 - \lambda_1\lambda_2\right)\left(e^{2Ax} - 2Ke^{Ax} + K^2\right) \Longrightarrow$$

$$-m\left(\lambda_1^2 - \lambda_1\lambda_2\right) = 0 \Longrightarrow \lambda_1 = \lambda_2 \text{ (contradiction with } \lambda_1 \neq \lambda_2)$$

Hence we don't have any solution in this case.

B). M is Ricci flat having a quarter-symmetric metric connection and $\dim F_i = 1$ for every $i \in \{1, ..., m\}$. Hence the system (3) becomes for every $i \in \{1, ..., m\}$:

$$\begin{cases} \sum\limits_{i=1}^{m}\left(\lambda\frac{f_i^{'}}{f_i}-\frac{f_i^{''}}{f_i}\right)=0 \\ -f_if_i^{''}+\left(\lambda f_i^2-f_if_i^{'}\right)\sum\limits_{j\neq i,\ j=1}^{m}\frac{f_j^{'}}{f_j}-(m-1)\lambda^2 f_i^2+m\lambda f_if_i^{'}=0 \end{cases} \tag{9}$$

Dividing by f_i^2 the second equation of (9) one can obtain the equivalent form:

$$\begin{cases} \sum\limits_{i=1}^{m}\left(\lambda\frac{f_i^{'}}{f_i}-\frac{f_i^{''}}{f_i}\right)=0 \\ -\frac{f_i^{''}}{f_i}+\left(\lambda-\frac{f_i^{'}}{f_i}\right)\sum\limits_{j\neq i,\ j=1}^{m}\frac{f_j^{'}}{f_j}-(m-1)\lambda^2+m\lambda\frac{f_i^{'}}{f_i}=0 \\ \qquad\qquad\qquad\text{for every } i\in\{1,...,m\} \end{cases} \tag{10}$$

We make now the following notations: $h_i=\frac{f_i^{'}}{f_i}$ for every $i\in\{1,...,m\}$ and $H=\sum\limits_{i=1}^{m}\frac{f_i^{'}}{f_i}=\sum\limits_{i=1}^{m}h_i$. We remark that $\frac{f_i^{''}}{f_i}=h_i^2+h_i^{'}$ for every $i\in\{1,...,m\}$. The system (10) becomes

$$\begin{cases} \sum\limits_{i=1}^{m}\left[\lambda_2 h_i-h_i^{'}-h_i^{2}\right]=0 \\ h_i^{'}=(h_i-\lambda)\left[(m-1)\lambda-H\right] \text{ for every } i\in\{1,...,m\} \end{cases} \tag{11}$$

Summing over i the above m equations of (11) we obtain $H^{'}=(H-m\lambda)\left[(m-1)\lambda-H\right]$. Thus:

1). If $H=(m-1)\lambda$, then $h_i^{'}=0$ which implies $h_i=c_i$ for every $i\in\{1,...,m\}$. From $\frac{f_i^{'}}{f_i}=h_i=c_i$ we obtain $f_i(x)=\eta_i e^{c_i x}$, with $\eta_i>0$ for every $i\in\{1,...,m\}$. Since $H=\sum\limits_{i=1}^{m}h_i$ we obtain that $\sum\limits_{i=1}^{m}c_i=(m-1)\lambda$. The first relation implies:

$$\sum_{i=1}^{m}\left[\lambda h_i-h_i^{'}-h_i^{2}\right]=0\Longrightarrow$$

$$\sum_{i=1}^{m}c_i^2=(m-1)\lambda^2$$

2). If $H=m\lambda$, then $h_i^{'}=-(h_i-\lambda)\lambda$ which implies $h_i(x)=\lambda+c_ie^{-\lambda x}$, with $c_i\in\mathbb{R}$ for every $i\in\{1,...,m\}$. From $H=\sum\limits_{i=1}^{m}h_i$ we obtain that $\sum\limits_{i=1}^{m}c_i=0$. Also the first relation implies:

$$\sum_{i=1}^{m}\left[\lambda h_i - h_i^{'} - h_i^{2}\right] = 0 \Longrightarrow$$

$$\left(-e^{-2\lambda x}\right)\sum_{i=1}^{m} c_i^2 = 0 \Longrightarrow c_i = 0 \text{ for every } i \in \{1,...,m\}$$

Hence $h_i(x) = \lambda$ and thus $f_i(x) = \eta_i e^{\lambda x}$, with $\eta_i > 0$ for every $i \in \{1,...,m\}$.

3). If $H \neq (m-1)\lambda$ and $H \neq m\lambda$, then $H^{'} = -H^2 + (2m-1)\lambda H - m(m-1)\lambda^2 = -\left[H - \frac{(2m-1)\lambda}{2}\right]^2 + \left(\frac{\lambda}{2}\right)^2$. Hence, taking into consideration the positive modulus, the case of negative modulus being similar, we obtain $H(x) = m\lambda + \frac{\lambda c}{e^{\lambda x}-c}$, with $c \in \mathbb{R}^*$.

Thus $h_i^{'} = (h_i - \lambda)\left(-\lambda - \frac{\lambda c}{e^{\lambda x}-c}\right)$ and we have $h_i(x) = \frac{\lambda e^{\lambda x}+c_i}{e^{\lambda x}-c}$, with $c_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$. From the relation $H = \sum_{i=1}^{m} h_i$ we obtain $\sum_{i=1}^{m} c_i = -(m-1)\lambda c$. Also the first relation implies:

$$\sum_{i=1}^{m}\left[\lambda h_i - h_i^{'} - h_i^{2}\right] = 0 \Longrightarrow$$

$$\sum_{i=1}^{m} c_i^2 = (m-1)\lambda^2 c^2$$

Hence $\frac{f_i^{'}}{f_i} = h_i$ implies $f_i(x) = \eta_i e^{-\frac{c_i}{c}}\left(e^{\lambda x} - c\right)^{1+\frac{c_i}{\lambda c}}$. Making the change of variable $\varepsilon_i = -\frac{c_i}{c}$ for every $i \in \{1,...,m\}$ we obtain the solution of the following form $f_i(x) = \eta_i e^{\varepsilon_i x}\left(e^{\lambda x} - c\right)^{1-\frac{\varepsilon_i}{\lambda}}$, with $\eta_i > 0$, $c \neq 0$, $c_i \in \mathbb{R}$ such that $\sum_{i=1}^{m} \varepsilon_i = -\frac{1}{c}\sum_{i=1}^{m} c_i = (m-1)\lambda$ and $\sum_{i=1}^{m} \varepsilon_i^2 = \frac{1}{c^2}\sum_{i=1}^{m} c_i^2 = (m-1)\lambda^2$. Also, making another change of variable $\delta_i = \lambda - \varepsilon_i$ for every $i \in \mathbb{R}$ we obtain $f_i(x) = \eta_i e^{(\lambda-\delta_i)x}\left(e^{\lambda x} - c\right)^{\frac{\delta_i}{\lambda}}$, with $\eta_i > 0$, $c \neq 0$, $\delta_i \in \mathbb{R}$ such that $\sum_{i=1}^{m} \delta_i = m\lambda - \sum_{i=1}^{m} \varepsilon_i = \lambda$ and $\sum_{i=1}^{m} \delta_i^2 = \sum_{i=1}^{m}\left(\lambda^2 - 2\lambda\varepsilon_i + \varepsilon_i^2\right) = m\lambda^2 - 2\lambda(m-1)\lambda + (m-1)\lambda^2 = \lambda^2$.

C). M is Ricci flat having a quarter-symmetric metric connection and all the warping functions are equal. Thus the system (3) becomes for every $i \in \{1,...,m\}$:

$$\begin{cases} \sum_{i=1}^{m} k_i\left(\lambda\frac{f^{'}}{f} - \frac{f^{''}}{f}\right) = 0 \\ \alpha_i - ff^{''} - (k_i-1)\left(f^{'}\right)^2 + \left(\lambda f^2 - ff^{'}\right)\frac{f^{'}}{f}(\overline{n} - 1 - k_i) + \\ \qquad +(2-\overline{n})\lambda^2 f^2 + (\overline{n} + k_i - 2)\lambda f f^{'} = 0 \end{cases} \tag{12}$$

We obtain that $\alpha_i = \alpha_0$ for every $i \in \{1, ..., m\}$ and the system (12) is equivalent to

$$\begin{cases} f'' = \lambda f' \\ \alpha_0 - ff'' + (2-\overline{n})\left(f'\right)^2 + (2\overline{n}-3)\lambda ff' + (2-\overline{n})\lambda^2 f^2 = 0 \end{cases} \tag{13}$$

which is also equivalent to

$$\begin{cases} f'' = \lambda f' \\ \alpha_0 + (2-\overline{n})\left(f' - \lambda f\right)^2 = 0 \end{cases} \tag{14}$$

Hence the solution is $f(x) = \frac{c}{\lambda}e^{\lambda x} + d$, with $c, d \in \mathbb{R}$ such that $\alpha_0 = (\overline{n}-2)\lambda^2 d^2$.

D). M is Ricci flat having a quarter-symmetric non-metric connection and all the warping functions are equal. Thus the system (2) becomes for every $i \in \{1, ..., m\}$:

$$\begin{cases} \sum_{i=1}^{m} k_i \left(\lambda_2 \frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1\lambda_2\right) = 0 \\ \alpha_i - ff'' - (k_i - 1)\left(f'\right)^2 + \left(\lambda f^2 - ff'\right)\frac{f'}{f}(\overline{n} - 1 - k_i) + \\ \qquad + \left[\lambda_2^2 + (1-\overline{n})\lambda_1\lambda_2\right] f^2 + \left[(\overline{n}-1)\lambda_1 + (k_i - 2)\lambda_2\right] ff' = 0 \end{cases} \tag{15}$$

which is equivalent to

$$\begin{cases} f'' - \lambda_2 f' - \left(\lambda_1^2 - \lambda_1\lambda_2\right) f = 0 \\ \alpha_0 - ff'' - (\overline{n}-2)\left(f'\right)^2 + \left[(\overline{n}-1)\lambda_1 + (\overline{n}-2)\lambda_2\right] ff' + \\ \qquad + \left[\lambda_2^2 + (1-\overline{n})\lambda_1\lambda_2\right] f^2 = 0 \end{cases} \tag{16}$$

where $\alpha_0 = \alpha_i$ for every $i \in \{1, ..., m\}$.

The associated equation is $t^2 - \lambda_2 t - \left(\lambda_1^2 - \lambda_1\lambda_2\right) = 0$ which has the discriminant $\Delta = (2\lambda_1 - \lambda_2)^2 \geq 0$.

I). If $\Delta = 0 \Longleftrightarrow 2\lambda_1 = \lambda_2$, then $t_{1,2} = \frac{\lambda_2}{2} = \lambda_1 = t$ and $f(x) = c_1 e^{tx} + c_2 x e^{tx}$, with $c_1, c_2 \in \mathbb{R}$ not both zero. Then the second equation of (16) becomes:

$$\alpha_0 + e^{2\lambda_1 x}\left[2\lambda_1^2 c_1^2 + (\overline{n}-3)\lambda_1 c_1 c_2 - (\overline{n}-2)c_2^2\right] +$$
$$xe^{2\lambda_1 x}\left[4\lambda_1^2 c_1 c_2 + (\overline{n}-3)\lambda_1 c_2^2\right] + x^2 e^{2\lambda_1 x}\left[2\lambda_1^2 c_2^2\right] = 0 \Longrightarrow$$

$$\begin{cases} \alpha_0 = 0 \\ 2\lambda_1^2 c_1^2 + (\overline{n}-3)\lambda_1 c_1 c_2 - (\overline{n}-2)c_2^2 = 0 \Longrightarrow c_1 = 0 \\ 4\lambda_1^2 c_1 c_2 + (\overline{n}-3)\lambda_1 c_2^2 = 0 \\ 2\lambda_1^2 c_2^2 = 0 \Longrightarrow c_2 = 0 \end{cases} \tag{17}$$

Hence we don't have any solution in this case.

II). If $\Delta > 0 \Longleftrightarrow 2\lambda_1 \neq \lambda_2$, then $t_1 = \lambda_1$, $t_2 = \lambda_2 - \lambda_1$, $t_1 \neq t_2$ and $f(x) = c_1 e^{t_1 x} + c_2 e^{t_2 x}$, with $c_1, c_2 \in \mathbb{R}$ not both zero. With $p = e^{t_1 x}$, and $q = e^{t_2 x}$, the second equation of (16) becomes:

$$\begin{gathered} \alpha_0 + \\ p^2\{-c_1^2 t_1^2 + (2-\overline{n})c_1^2 t_1^2 + [(\overline{n}-2)\lambda_2 + (\overline{n}-1)\lambda_1]\, c_1^2 t_1 + [\lambda_2^2 + (1-\overline{n})\lambda_1\lambda_2] c_1^2\} + \\ q^2\{-c_2^2 t_2^2 + (2-\overline{n})c_2^2 t_2^2 + [(\overline{n}-2)\lambda_2 + (\overline{n}-1)\lambda_1]\, c_2^2 t_2 + [\lambda_2^2 + (1-\overline{n})\lambda_1\lambda_2] c_2^2\} + \\ pq\{-c_1 c_2 t_1^2 - c_1 c_2 t_2^2 + (4-2\overline{n})c_1 c_2 t_1 t_2 + [(\overline{n}-2)\lambda_2 + (\overline{n}-1)\lambda_1]\, c_1 c_2\, (t_1 + t_2) + \\ 2\left[\lambda_2^2 + (1-\overline{n})\lambda_1\lambda_2\right] c_1 c_2\} = 0 \Longrightarrow \end{gathered}$$

$$\begin{cases} \alpha_0 = 0 \\ -(\overline{n}-1)c_1^2 t_1^2 + [(\overline{n}-2)\lambda_2 + (\overline{n}-1)\lambda_1]\, c_1^2 t_1 + \left[\lambda_2^2 + (1-\overline{n})\lambda_1\lambda_2\right] c_1^2 = 0 \\ -(\overline{n}-1)c_2^2 t_2^2 + [(\overline{n}-2)\lambda_2 + (\overline{n}-1)\lambda_1]\, c_2^2 t_2 + \left[\lambda_2^2 + (1-\overline{n})\lambda_1\lambda_2\right] c_2^2 = 0 \\ -c_1 c_2 \left(t_1^2 + t_2^2\right) - (2\overline{n}-4)c_1 c_2 t_1 t_2 + \\ \qquad [(\overline{n}-2)\lambda_2 + (\overline{n}-1)\lambda_1]\, c_1 c_2\, (t_1 + t_2) + \\ \qquad 2\left[\lambda_2^2 + (1-\overline{n})\lambda_1\lambda_2\right] c_1 c_2 = 0 \end{cases} \tag{18}$$

We obtain that $c_1 = 0$ and the solution is $f(x) = c_2 e^{(\lambda_2 - \lambda_1)x}$, with $\alpha_0 = 0$, $c_2 > 0$ and $(2\overline{n}-2)\lambda_1 = \overline{n}\lambda_2$.

E). M is Ricci flat with a semi-symmetric non-metric connection and it has the supplementary condition $H = \sum\limits_{i=1}^{m} k_i \frac{f_i'}{f_i} = 0$. Then the system (1) becomes:

$$\begin{cases} \sum\limits_{i=1}^{m} k_i \frac{f_i'}{f_i} = 0 \\ \sum\limits_{i=1}^{m} k_i \frac{f_i''}{f_i} = \overline{n} - 1 \\ \lambda_i - f_i f_i'' + \left(f_i'\right)^2 = 0 \text{ for every } i \in \{1, ..., m\} \end{cases} \tag{19}$$

Integrating the third equation of (19) using the change of variable $f_i' = g_i(f_i)$, after the computations, one can obtain the solution of the following form: $f_i(x) = \alpha_i e^{\varepsilon_i x} + \beta_i e^{-\varepsilon_i x}$, with $\alpha_i, \beta_i \in \mathbb{R}$ and $\varepsilon_i > 0$ for every $i \in \{1, ..., m\}$. Putting that in the second and the third equations of (19) one can obtain $\lambda_i = 4\alpha_i\beta_i\varepsilon_i^2$ for every $i \in \{1, ..., m\}$ and $\sum\limits_{i=1}^{m} k_i \varepsilon_i^2 = \overline{n} - 1$. Let be Ω the following property: $\left[\underset{0 \leq p, r \leq m \text{ such that } p+r=m}{i_1, ..., i_p, j_1, ..., j_r \in \{1, ..., m\} \text{ different}}\right]$. Then, the first equation of (19) implies:

$$\sum_{i=1}^{m} k_i\varepsilon_i \frac{\alpha_i e^{2\varepsilon_i x} - \beta_i}{\alpha_i e^{2\varepsilon_i x} + \beta_i} = 0 \Longrightarrow$$

$$\sum_{\Omega} \left(\gamma_{i_1} + ... + \gamma_{i_p} - \gamma_{j_1} - ... - \gamma_{j_r}\right) \alpha_{i_1}...\alpha_{i_p}\beta_{j_1}...\beta_{j_r} e^{\left(\delta_{i_1}+...+\delta_{i_p}-\delta_{j_1}-...-\delta_{j_r}\right)x} = 0$$

where $\gamma_i = k_i\varepsilon_i > 0$ and $\delta_i = 2\varepsilon_i > 0$ for every $i \in \{1, ..., m\}$.

We suppose that for every $(i_1, ..., i_p, j_1, ..., j_r) \neq \left(i_1^{'}, ..., i_p^{'}, j_1^{'}, ..., j_r^{'}\right)$, with $i_1, ..., i_p, j_1, ..., j_r \in \{1, ..., m\}$ different and $i_1^{'}, ..., i_p^{'}, j_1^{'}, ..., j_r^{'} \in \{1, ..., m\}$ different, $0 \leq p, r \leq m$ such that $p + r = m$ we have $\delta_{i_1} + ... + \delta_{i_p} - \delta_{j_1} - ... - \delta_{j_r} \neq \delta_{i_1^{'}} + ... + \delta_{i_p^{'}} - \delta_{j_1^{'}} - ... - \delta_{j_r^{'}}$. Hence from the above equation we obtain the 2^m relations of the form: $\left(\gamma_{i_1} + ... + \gamma_{i_p} - \gamma_{j_1} - ... - \gamma_{j_r}\right) \alpha_{i_1}...\alpha_{i_p}\beta_{j_1}...\beta_{j_r} = 0$ for every $i_1, ..., i_p, j_1, ..., j_r \in \{1, ..., m\}$ different with $0 \leq p, r \leq m$ such that $p + r = m$.

I). Suppose that $\pm\gamma_1 \pm ... \pm \gamma_m \neq 0$ for all the combinations of '+' and '-', except the cases when all the combinations have the sign '+' or all the combinations have the sign '-'. After a renumbering of the α_i's and β_i's, one can obtain the following:

$$\left\{\begin{array}{l} \alpha_1\beta_2...\beta_m = 0 \Longrightarrow \alpha_1 = 0 \\ \beta_1\alpha_2\beta_3...\beta_m = 0 \Longrightarrow \alpha_2 = 0 \\ .... \\ \beta_1\beta_2...\beta_{m-1}\alpha_m = 0 \Longrightarrow \alpha_m = 0 \\ \beta_1\beta_2\beta_3...\beta_m = 0 \Longrightarrow \beta_1 = 0 \end{array}\right. \Longrightarrow f_1(x) = 0 \text{ (contradiction)}$$

The other cases are analogous. Hence, we don't have any solution in this case.

II). Thus there exists $\gamma_{i_1} + ... + \gamma_{i_p} - \gamma_{j_1} - ... - \gamma_{j_r} = 0 \Longleftrightarrow -\gamma_{i_1} - ... - \gamma_{i_p} + \gamma_{j_1} + ... + \gamma_{j_r} = 0$ and without the loss of generality we can consider $\pm\gamma_1 \pm ... \pm \gamma_m - K \neq 0$ for the other combinations different from $i_1, ..., i_p, j_1, ..., j_r$. We can suppose $i_1 = 1, ..., i_p = p$, $j_1 = p + 1, ..., j_m = m$, the other cases being similar, and after a renumbering of the α_i's and β_i's, one can obtain the following: $\alpha_1\alpha_2...\alpha_m = 0 \Longrightarrow \alpha_1 = 0$, $\beta_1\alpha_2...\alpha_m = 0 \Longrightarrow \alpha_2 = 0$,..., $\beta_1...\beta_{p-1}\alpha_p...\alpha_m = 0 \Longrightarrow \alpha_p = 0$. Suppose that $\beta_{p+1} \neq 0$. Then from $\beta_1...\beta_{p+1}\alpha_{p+2}...\alpha_m = 0 \Longrightarrow \alpha_{p+2} = 0$, $\beta_1...\beta_{p+2}\alpha_{p+3}...\alpha_m = 0 \Longrightarrow \alpha_{p+3} = 0, ..., \beta_1...\beta_{m-1}\alpha_m = 0 \Longrightarrow \alpha_m = 0$ and from $\beta_1...\beta_m = 0 \Longrightarrow \beta_i = 0$ for $i \in \{1, ..., m\}$. Hence the function $f_i = 0$ which is a contradiction. Thus $\beta_{p+1} = 0$ and from $\beta_1...\beta_p\alpha_{p+1}\beta_{p+2}...\beta_m = 0 \Longrightarrow \beta_{p+2} = 0$, $\beta_1...\beta_p\alpha_{p+1}\alpha_{p+2}\beta_{p+3}...\beta_m = 0 \Longrightarrow \beta_{p+3} = 0, ..., \beta_1...\beta_p\alpha_{p+1}...\alpha_{m-1}\beta_m = 0 \Longrightarrow \beta_m = 0$.

Hence we obtain the following solution: $f_1(x) = \beta_1 e^{-\varepsilon_1 x}, ..., f_p(x) = \beta_p e^{-\varepsilon_p x}$, $f_{p+1}(x) = \alpha_{p+1}e^{\varepsilon_{p+1}x}, ..., f_m(x) = \alpha_m e^{\varepsilon_m x}$, where $\beta_i > 0$ for every $i \in \{1, ..., p\}$ and $\alpha_j > 0$ for every $j \in \{p + 1, ..., m\}$, with $1 \leq p \leq m - 1$. The other cases are analogous. Thus, taking into consideration all the cases, we deduce that the solution can be written as: $f_i(x) = \eta_i e^{\varepsilon_i x}$, where $\eta_i > 0$, $\varepsilon_i \neq 0$, $\lambda_i = 0$ for every $i \in \{1, ..., m\}$ such that $\sum_{i=1}^{m} k_i\varepsilon_i = 0$ and $\sum_{i=1}^{m} k_i\varepsilon_i^2 = \overline{n} - 1$.

F). M is Ricci flat with a quarter-symmetric non-metric connection and it has the supplementary condition $H = \sum_{i=1}^{m} k_i \frac{f_i^{'}}{f_i} = (\overline{n}-1)\lambda_1 - \lambda_2$. Then the system (2) becomes:

$$\begin{cases} \sum_{i=1}^{m} k_i \frac{f_i^{'}}{f_i} = (\overline{n}-1)\lambda_1 - \lambda_2 \\ \sum_{i=1}^{m} k_i \frac{f_i^{''}}{f_i} = (\overline{n}-1)\lambda_1^2 - \lambda_2^2 \\ \lambda_i - f_i f_i^{''} + \left(f_i^{'}\right)^2 = 0 \text{ for every } i \in \{1, ..., m\}. \end{cases} \quad (20)$$

We denote by $K = (\overline{n}-1)\lambda_1 - \lambda_2$ and remark that $K \neq 0$. Hence, integrating the third equation of (20) using the change of variable $f_i^{'} = g_i(f_i)$, after the computations, one can obtain the solution of the following form: $f_i(x) = \alpha_i e^{\varepsilon_i x} + \beta_i e^{-\varepsilon_i x}$, with $\alpha_i, \beta_i \in \mathbb{R}$ and $\varepsilon_i > 0$ for every $i \in \{1, ..., m\}$. Putting that in the second and the third equations of (20) one can obtain $\lambda_i = 4\alpha_i\beta_i\varepsilon_i^2$ for every $i \in \{1, ..., m\}$ and $\sum_{i=1}^{m} k_i \varepsilon_i^2 = (\overline{n}-1)\lambda_1^2 - \lambda_2^2$. Also the first equation of (20) implies:

$$\sum_{i=1}^{m} k_i \varepsilon_i \frac{\alpha_i e^{2\varepsilon_i x} - \beta_i}{\alpha_i e^{2\varepsilon_i x} + \beta_i} = K \Longrightarrow$$

$$\sum_{\Omega} \left(\gamma_{i_1} + ... + \gamma_{i_p} - \gamma_{j_1} - ... - \gamma_{j_r} - K\right) \alpha_{i_1} ... \alpha_{i_p} \beta_{j_1} ... \beta_{j_r} e^{\left(\delta_{i_1} + ... + \delta_{i_p} - \delta_{j_1} - ... - \delta_{j_r}\right)x} = 0,$$

where $\gamma_i = k_i \varepsilon_i > 0$ and $\delta_i = 2\varepsilon_i > 0$ for every $i \in \{1, ..., m\}$.

We suppose that for every $(i_1, ..., i_p, j_1, ..., j_r) \neq \left(i_1^{'}, ..., i_p^{'}, j_1^{'}, ..., j_r^{'}\right)$, with $i_1, ..., i_p, j_1, ..., j_r \in \{1, ..., m\}$ different and $i_1^{'}, ..., i_p^{'}, j_1^{'}, ..., j_r^{'} \in \{1, ..., m\}$ different, $0 \leq p, r \leq m$ such that $p + r = m$ we have $\delta_{i_1} + ... + \delta_{i_p} - \delta_{j_1} - ... - \delta_{j_r} \neq \delta_{i_1^{'}} + ... + \delta_{i_p^{'}} - \delta_{j_1^{'}} - ... - \delta_{j_r^{'}}$. Hence from the above equation we obtain the 2^m relations of the form: $\left(\gamma_{i_1} + ... + \gamma_{i_p} - \gamma_{j_1} - ... - \gamma_{j_r} - K\right) \alpha_{i_1} ... \alpha_{i_p} \beta_{j_1} ... \beta_{j_r} = 0$ for every $i_1, ..., i_p, j_1, ..., j_r \in \{1, ..., m\}$ different with $0 \leq p, r \leq m$ such that $p + r = m$.

I). Suppose that $\pm\gamma_1 \pm ... \pm \gamma_m - K \neq 0$ for all the combinations of '+' and '-'. After a renumbering of the α_i's and β_i's, one can obtain the following:

$$\begin{cases} \alpha_1 \beta_2 ... \beta_m = 0 \Longrightarrow \alpha_1 = 0 \\ \beta_1 \alpha_2 \beta_3 ... \beta_m = 0 \Longrightarrow \alpha_2 = 0 \\ .... \\ \beta_1 \beta_2 ... \beta_{m-1} \alpha_m = 0 \Longrightarrow \alpha_m = 0 \\ \beta_1 \beta_2 \beta_3 ... \beta_m = 0 \Longrightarrow \beta_1 = 0 \end{cases} \Longrightarrow f_1(x) = 0 \text{ (contradiction)}$$

The other cases are analogous. Hence, we don't have any solution in this case.

II). Thus there exists $\gamma_{i_1}+...+\gamma_{i_p}-\gamma_{j_1}-...-\gamma_{j_r}-K=0 \Longleftrightarrow -\gamma_{i_1}-...-\gamma_{i_p}+\gamma_{j_1}+...+\gamma_{j_r}+K=0$ and without the loss of generality we can consider $\pm\gamma_1 \pm ... \pm \gamma_m - K \neq 0$ for the other combinations different from $i_1, ..., i_p, j_1, ..., j_r$. We can suppose $i_1 = 1, ..., i_p = p$, $j_1 = p+1, ..., j_m = m$, the other cases being similar, and after a renumbering of the α_i's and β_i's, one can obtain the following: $\alpha_1\alpha_2...\alpha_m = 0 \Longrightarrow \alpha_1 = 0$, $\beta_1\alpha_2...\alpha_m = 0 \Longrightarrow \alpha_2 = 0$,..., $\beta_1...\beta_{p-1}\alpha_p...\alpha_m = 0 \Longrightarrow \alpha_p = 0$. Suppose that $\beta_{p+1} \neq 0$. Then from $\beta_1...\beta_{p+1}\alpha_{p+2}...\alpha_m = 0 \Longrightarrow \alpha_{p+2} = 0$, $\beta_1...\beta_{p+2}\alpha_{p+3}...\alpha_m = 0 \Longrightarrow \alpha_{p+3} = 0, ...,$ $\beta_1...\beta_{m-1}\alpha_m = 0 \Longrightarrow \alpha_m = 0$ and from $\beta_1...\beta_m = 0 \Longrightarrow \beta_i = 0$ for $i \in \{1, ..., m\}$. Hence the function $f_i = 0$ which is a contradiction. Thus $\beta_{p+1} = 0$ and from $\beta_1...\beta_p\alpha_{p+1}\beta_{p+2}...\beta_m = 0 \Longrightarrow \beta_{p+2} = 0$, $\beta_1...\beta_p\alpha_{p+1}\alpha_{p+2}\beta_{p+3}...\beta_m = 0 \Longrightarrow \beta_{p+3} = 0, ...,$ $\beta_1...\beta_p\alpha_{p+1}...\alpha_{m-1}\beta_m = 0 \Longrightarrow \beta_m = 0$.

Hence we obtain the following solution: $f_1(x) = \beta_1 e^{-\varepsilon_1 x}, ..., f_p(x) = \beta_p e^{-\varepsilon_p x}$, $f_{p+1}(x) = \alpha_{p+1} e^{\varepsilon_{p+1} x}, ..., f_m(x) = \alpha_m e^{\varepsilon_m x}$, where $\beta_i > 0$ for every $i \in \{1, ..., p\}$ and $\alpha_j > 0$ for every $j \in \{p+1, ..., m\}$, with $1 \leq p \leq m-1$. The other cases are analogous. Thus, taking into consideration all the cases, we deduce that the solution can be written as: $f_i(x) = \eta_i e^{\varepsilon_i x}$, where $\eta_i > 0$, $\varepsilon_i \neq 0$, $\lambda_i = 0$ for every $i \in \{1, ..., m\}$ such that $\sum\limits_{i=1}^{m} k_i\varepsilon_i = K = (\overline{n}-1)\lambda_1 - \lambda_2$ and $\sum\limits_{i=1}^{m} k_i\varepsilon_i^2 = (\overline{n}-1)\lambda_1^2 - \lambda_2^2$.

Open problem. Similar to the point E), we can consider the case when M is Ricci flat with a quarter-symmetric non-metric connection having the supplementary condition $H = \sum\limits_{i=1}^{m} k_i \frac{f_i'}{f_i} = 0$. Then the system (2) implies more difficult equations for every $i \in \{1, ..., m\}$:

$$\begin{cases} \sum\limits_{i=1}^{m} k_i \frac{f_i'}{f_i} = 0 \\ \sum\limits_{i=1}^{m} k_i \frac{f_i''}{f_i} = (\overline{n}-1)\left(\lambda_1^2 - \lambda_1\lambda_2\right) \\ \lambda_i - f_i f_i'' + \left(f_i'\right)^2 - \lambda_2\left[(\overline{n}-1)\lambda_1 - \lambda_2\right] f_i^2 + \left[(\overline{n}-1)\lambda_1 - \lambda_2\right] f_i' f_i = 0 \end{cases} \quad (21)$$

Find, if there exists, a particular solution of the system (21) or prove it has no solution.

References

1. A. L. Besse, Einstein Manifolds, Springer-Verlag, Reprint of 1987 Edition, pages 274.
2. R. L. Bishop, B. O'Neill, *Manifolds of Negative Curvature*, "Trans. Amer. Math. Soc.", **145** (1969), pp. 1-49.
3. F. Dobarro, B. Unal, *Curvature of Multiply Warped Products*, "Journal of Geometry and Physics", **55**(1), September 2005, pp. 75-106.

4. D. Dumitru, On Einstein Spaces of Odd Dimension, "Buletinul Universităţii Transilvania", Braşov, **14** (49), 2007, pp. 95-97.
5. D. Dumitru, *On Multiple Generalized Robertson-Walker Space-Time*, "Analele Univ. Oradea", Fasc. Matematică, Tom **XVIII,** 2011, pp. 159-166.
6. D. Dumitru, *On Multiply Einstein Warped Products*, "Analele Şt. Univ. Al. I. Cuza", Iaşi, (S.N.), Seria Matematică, Tom **LXII**, f.1, 2016, pp. 213-220.
7. M. M. Tripathi, *A New Connection in a Riemannian Manifold*, "International Electronic Journal of Geometry", **1**(1), 2008, pp. 15-24.
8. Y. Wang, *Curvature of Multiply Warped Products with an Affine Connection*, "Bull. Korean Math. Soc.", **50**(5), 2013, pp. 1567-1586.
9. Y. Wang, *Multiply Warped Products with a Semi-Symmetric Metric Connection*, "Abstr. Appl. Anal.", 2014, Article ID 742371, 12 pages.
10. Q. Qu, Y. Wang, *Multiply Warped Products with a Quarter-Symmetric Connection*, "J. Math. Anal. Appl.", **431** (2), 2015, pp. 955-987.