Particular Cases of Special Robertson-Walker Multiply Warped Products with a Semi-Symmetric Connection[1]

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Abstract

In this paper we study some particular cases of Einstein equations for special Robertson-Walker multiply warped products $M = I \times_{f_1} F_1 \times \ldots \times_{f_m} F_m$*, where* $I \subset \mathbb{R}$ *is an open interval,* $\dim I = 1$, $f_i : I \longrightarrow (0, \infty)$, $f_i \in \mathcal{C}^\infty(I)$, $\dim F_i = k_i \geq 1$ *for every* $i \in \{1, ..., m\}$, $m \geq 1$, *having an affine semi-symmetric connection. We compute the warping functions for* M *Ricci flat and equal warping functions and* M *Ricci flat and all the fibres having the dimensions equal to* 1.

Key words. *Einstein space, multiply warped product, warping function, semi-symmetric connection.*

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1. ***Introduction***

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection on M.

Definition 1.1. ([6], [7], [8]) a). A linear connection $\overline{\nabla}$ on M is called a *semi-symmetric connection* if the torsion tensor T of $\overline{\nabla}$, $T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]$, satisfies $T(X,Y) = \pi(Y)X - \pi(X)Y$, where π is a 1-form associated with the vector field P on M defined by $\pi(X) = g(X,P)$.

b). $\overline{\nabla}$ is called a *semi-symmetric metric connection* if $\overline{\nabla} g = 0$.

c). $\overline{\nabla}$ is called a *semi-symmetric non-metric connection* if it satisfies $\overline{\nabla} g \neq 0$.

We have the following two well-known results:

Theorem 1.1. *(theorem 3.1, p. 7, [7], semi-symmetric non-metric connection) Let* $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times \ldots \times_{f_m} F_m$ *be a multiply warped product with the metric tensor* $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus \ldots \oplus f_m^2 g_{F_m}$ *and* $P = \left(\frac{\partial}{\partial t}\right)$. *Then* $(M, \overline{\nabla})$ *is Einstein with the Einstein constant* λ *if and only if the following conditions are satisfied for every* $i \in \{1, ..., m\}$:

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$$\begin{cases} \left(F_i, \nabla^{F_i}\right) \text{ is Einstein with Einstein constant } \lambda_i \\ \sum\limits_{i=1}^{m} k_i \left(1 - \frac{f_i''}{f_i}\right) = \lambda \\ \lambda_i - f_i f_i'' - (k_i - 1)\left(f_i'\right)^2 - f_i f_i' \sum\limits_{j \neq i,\, j=1}^{m} k_j \frac{f_j'}{f_j} + f_i^2 \sum\limits_{j=1}^{m} k_j \frac{f_j'}{f_j} = \lambda f_i^2 \end{cases} \tag{1}$$

Theorem 1.2. *(theorem 15, p. 5, [8], semi-symmetric metric connection) Let $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times ... \times_{f_m} F_m$ be a multiply warped product with the metric tensor $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus ... \oplus f_m^2 g_{F_m}$ and $P = \left(\frac{\partial}{\partial t}\right)$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant λ if and only if the following conditions are satisfied for every $i \in \{1, ..., m\}$*

$$\begin{cases} \left(F_i, \nabla^{F_i}\right) \text{ is Einstein with Einstein constant } \lambda_i \\ \sum\limits_{i=1}^{m} k_i \left(\frac{f_i'}{f_i} - \frac{f_i''}{f_i}\right) = \lambda \\ \lambda_i - f_i f_i'' - (k_i - 1)(f_i')^2 + (f_i^2 - f_i f_i') \sum\limits_{j \neq i,\, j=1}^{m} k_j \frac{f_j'}{f_j} + \\ \qquad + (2 - \overline{n}) f_i^2 + (\overline{n} + k_i - 2) f_i f_i' = \lambda f_i^2 \end{cases} \tag{2}$$

where $\overline{n} = 1 + \sum\limits_{i=1}^{m} k_i$.

2. *Main results*

The aim of this article is to solve the systems (1) and (2) for some particular cases. For the beginning we consider the case of Robertson-Walker multiply warped products with a semi-symmetric non-metric connection.

I). M is Ricci flat and all the warping functions are equal in the system (1). We denote $f_i = f$ for every $i \in \{1, ..., m\}$, $m \geq 1$ and we obtain

$$\begin{cases} \sum\limits_{i=1}^{m} k_i \left(1 - \frac{f''}{f}\right) = 0 \\ \lambda_i - f f'' - (k_i - 1)\left(f'\right)^2 - f f' \sum\limits_{j \neq i,\, j=1}^{m} k_j \frac{f'}{f} + f^2 \sum\limits_{j=1}^{m} k_j \frac{f'}{f} = 0 \end{cases} \tag{3}$$

We obtain that $\lambda_i = \lambda_0$ for every $i \in \{1, ..., m\}$ and the system (3) becomes

$$\begin{cases} \left(\sum\limits_{i=1}^{m} k_i\right)\left(1 - \frac{f''}{f}\right) = 0 \\ \lambda_0 - f f'' + \left(f'\right)^2 \left(1 - \sum\limits_{j=1}^{m} k_j\right) + f f' \left(\sum\limits_{j=1}^{m} k_j\right) = 0 \end{cases} \tag{4}$$

From the first equation we obtain that $1-\frac{f''}{f}=0 \Longrightarrow f''=f \Longrightarrow f(x)=c_1e^x+c_2e^{-x}$. Hence the second equation becomes $\lambda_0-(c_1^2e^{2x}+c_2^2e^{-2x}+2c_1c_2)+$ $+(c_1^2e^{2x}+c_2^2e^{-2x}-2c_1c_2)(1-\sum_{j=1}^{m}k_j)+ +(c_1^2e^{2x}-c_2^2e^{-2x})(\sum_{j=1}^{m}k_j)=0 \Longleftrightarrow$

$$\begin{cases} \lambda_0-2c_1c_2-2c_1c_2\left(1-\sum_{j=1}^{m}k_j\right)=0 \Longrightarrow \lambda_0=0 \\ -c_1^2+c_1^2\left(1-\sum_{j=1}^{m}k_j\right)+c_1^2\left(\sum_{j=1}^{m}k_j\right)=0 \text{ (true)} \\ -c_2^2+c_2^2\left(1-\sum_{j=1}^{m}k_j\right)-c_2^2\left(\sum_{j=1}^{m}k_j\right)=0 \Longrightarrow c_2=0 \end{cases} \tag{5}$$

Thus $\lambda_0=0$ and $f(x)=ce^x$ with $c>0$.

II). M is Ricci flat and $M=I\times_{f_1}F_1$ with $\dim F_1=1$. Then

$$\begin{cases} 1-\frac{f_1''}{f_1}=0 \\ -f_1f_1''+f_1^2\cdot\frac{f_1'}{f_1}=0 \end{cases} \tag{6}$$

We obtain that $-f_1^2+f_1f_1'=0$ which implies $f_1(x)=ce^x$, with $c>0$.

III). M is Ricci flat and $M=I\times_{f_1}F_1\times_{f_2}F_2$ with $\dim F_1=\dim F_2=1$. Then

$$\begin{cases} 1-\frac{f_1''}{f_1}+1-\frac{f_2''}{f_2}=0 \\ -f_1f_1''-f_1f_1'\cdot\frac{f_2'}{f_2}+f_1^2\left(\frac{f_1'}{f_1}+\frac{f_2'}{f_2}\right)=0 \\ -f_2f_2''-f_2f_2'\cdot\frac{f_1'}{f_1}+f_2^2\left(\frac{f_1'}{f_1}+\frac{f_2'}{f_2}\right)=0 \end{cases} \tag{7}$$

The system (7) is equivalent to

$$\begin{cases} \frac{f_1''}{f_1}+\frac{f_2''}{f_2}=2 \\ -\frac{f_1''}{f_1}-\frac{f_1'f_2'}{f_1f_2}+\left(\frac{f_1'}{f_1}+\frac{f_2'}{f_2}\right)=0 \\ -\frac{f_2''}{f_2}-\frac{f_1'f_2'}{f_1f_2}+\left(\frac{f_1'}{f_1}+\frac{f_2'}{f_2}\right)=0 \end{cases} \tag{8}$$

From the second equation we obtain that $\frac{f_1''}{f_1}=\frac{f_2''}{f_2}$ which together with the first equation implies $\frac{f_1''}{f_1}=\frac{f_2''}{f_2}=1$.

Thus the second of the third equation becomes $-1-\frac{f_1'f_2'}{f_1f_2}+\frac{f_1'}{f_1}+\frac{f_2'}{f_2}=0 \Longleftrightarrow \left(\frac{f_1'}{f_1}-1\right)\left(1-\frac{f_2'}{f_2}\right)=0$. Hence a). $\frac{f_1'}{f_1} = 1$ and thus $f_1(x) = ce^x$ and $f_2(x) = c_1e^x + c_2e^{-x}$, with $c, c_1, c_2 > 0$.

b). $\frac{f_2'}{f_2} = 1$ and thus $f_2(x) = ce^x$ and $f_1(x) = c_1e^x + c_2e^{-x}$, with $c, c_1, c_2 > 0$.

IV). M is Ricci flat and $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times ... \times_{f_m} F_m$ with $\dim F_i = 1$ for every $i \in \{1, ..., m\}$ and $m \geq 3$. Then for every $i \in \{1, ..., m\}$ the system (1) becomes

$$\begin{cases} \sum\limits_{i=1}^{m} \left(1 - \frac{f_i^{''}}{f_i}\right) = 0 \\ -f_i f_i^{''} - f_i f_i^{'} \sum\limits_{j\neq i,\ j=1}^{m} \frac{f_j^{'}}{f_j} + f_i^2 \sum\limits_{j=1}^{m} \frac{f_j^{'}}{f_j} = 0 \end{cases} \tag{9}$$

Dividing by f_i^2 the second equation of (9) one can obtain for every $i \in \{1, ..., m\}$ the equivalent form

$$\begin{cases} \sum\limits_{i=1}^{m} \left(1 - \frac{f_i^{''}}{f_i}\right) = 0 \\ -\frac{f_i^{''}}{f_i} - \frac{f_i^{'}}{f_i} \sum\limits_{j\neq i,\ j=1}^{m} \frac{f_j^{'}}{f_j} + \sum\limits_{j=1}^{m} \frac{f_j^{'}}{f_j} = 0 \end{cases} \tag{10}$$

We make now the following notations: $h_i = \frac{f_i^{'}}{f_i}$ for every $i \in \{1, ..., m\}$ and $H = \sum\limits_{i=1}^{m} \frac{f_i^{'}}{f_i} = \sum\limits_{i=1}^{m} h_i$. We remark that $\frac{f_i^{''}}{f_i} = h_i^2 + h_i^{'}$ for every $i \in \{1, ..., m\}$ and the system (10) becomes:

$$\begin{cases} \sum\limits_{i=1}^{m} \left[1 - h_i^2 - h_i^{'}\right] = 0 \\ h_i^{'} = (-H)h_i + H \text{ for every } i \in \{1, ..., m\} \end{cases} \tag{11}$$

Summing over i the above m equations of (11) we obtain the equation $H^{'} = mH - H^2$.

i). If $H = 0$, then $h_i^{'} = 0$ implies $h_i = \varepsilon_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$. From $H = \sum\limits_{i=1}^{m} h_i$ we obtain $\sum\limits_{i=1}^{m} \varepsilon_i = 0$ and from $\sum\limits_{i=1}^{m} \left[1 - h_i^2 - h_i^{'}\right] = 0$ we obtain $\sum\limits_{i=1}^{m} \varepsilon_i^2 = m$. Hence $\frac{f_i^{'}}{f_i} = h_i = \varepsilon_i$ implies $f_i(x) = \eta_i e^{\varepsilon_i x}$, with $\eta_i > 0$, $\varepsilon_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$ such that $\sum\limits_{i=1}^{m} \varepsilon_i = 0$ and $\sum\limits_{i=1}^{m} \varepsilon_i^2 = m \geq 3$.

ii). If $H = m$, then $h_i^{'} = (-H)h_i + H$ has the solution $h_i(x) = \frac{e^{mx} + \varepsilon_i}{e^{mx}}$, with $\varepsilon_i \in \mathbb{R}$. From $H = \sum\limits_{i=1}^{m} h_i$ we obtain $\sum\limits_{i=1}^{m} \varepsilon_i = 0$.

Also from $\sum\limits_{i=1}^{m} \left[1 - h_i^2 - h_i^{'}\right] = 0$ we obtain $\sum\limits_{i=1}^{m} \varepsilon_i^2 = 0$ which implies $\varepsilon_i = 0$ for every $i \in \{1, ..., m\}$.

Hence $\frac{f_i^{'}}{f_i} = h_i = 1$ implies $f_i(x) = \eta_i e^x$, with $\eta_i > 0$ for every $i \in \{1, ..., m\}$.

iii). If $H \neq 0$ and $H \neq m$, then we obtain the solution $H(x) = \frac{me^{mx}}{me^{mx}+c}$, where $c \in \mathbb{R}$. Moreover $h_i^{'} = (-H)\, h_i + H$ implies $h_i(x) = \frac{e^{mx}+\varepsilon_i}{e^{mx}+c}$, where $\varepsilon_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$. We remark that since $H = \sum\limits_{i=1}^{m} h_i$ we obtain $\sum\limits_{i=1}^{m} \varepsilon_i = 0$. We also remark that $\sum\limits_{i=1}^{m} \left[1 - h_i^2 - h_i^{'}\right] = 0 \Longleftrightarrow \sum\limits_{i=1}^{m} \left(h_i^2 + h_i^{'}\right) = m$ and we obtain $\left(m^2 c - 2mc\right) e^{mx} + \left(\sum\limits_{i=1}^{m} \varepsilon_i^2 - mc^2\right) = 0$ which implies

$$\begin{cases} m^2 c = 2mc \\ \sum\limits_{i=1}^{m} \varepsilon_i^2 = mc^2 \end{cases} \tag{12}$$

The system (12) has the solutions $c = 0$ and $\varepsilon_i = 0$ for every $i \in \{1, ..., m\}$ since $m \geq 3$. Thus we obtain $H(x) = m$ which is a contradiction.

We consider now the case of Robertson-Walker multiply warped products with a semi-symmetric metric connection.

V). M is Ricci flat and all the warping functions are equal in the system (2). We denote $f_i = f$ and we obtain for every $i \in \{1, ..., m\}$, $m \geq 1$ that

$$\begin{cases} \sum\limits_{i=1}^{m} k_i \left(\frac{f^{'}}{f} - \frac{f^{''}}{f}\right) = 0 \\ \lambda_i - ff^{''} - (k_i - 1)\left(f^{'}\right)^2 + \left(f^2 - ff^{'}\right) \sum\limits_{j \neq i,\, j=1}^{m} k_j \frac{f^{'}}{f} + \\ \qquad + (1 - m) f^2 + (m + k_i - 1) ff^{'} = 0 \end{cases} \tag{13}$$

We obtain that $\lambda_i = \lambda_0$ for every $i \in \{1, ..., m\}$ and the system (13) becomes

$$\begin{cases} \left(\sum\limits_{i=1}^{m} k_i\right)\left(\frac{f^{'}}{f} - \frac{f^{''}}{f}\right) = 0 \\ \lambda_0 - ff^{''} + \left(f^{'}\right)^2 \left(1 - \sum\limits_{j=1}^{m} k_j\right) + ff^{'} \left(\sum\limits_{j=1}^{m} k_j + m - 1\right) + \\ \qquad + (1 - m) f^2 = 0 \end{cases} \tag{14}$$

From the first equation we obtain $f^{'} = f^{''}$ which gives $f(x) = ce^x$, where $c > 0$. Hence the second equation becomes

$$\lambda_0 - c^2 e^{2x} + \left(1 - \sum\limits_{j=1}^{m} k_j\right) c^2 e^{2x} + \left(m - 1 + \sum\limits_{j=1}^{m} k_j\right) c^2 e^{2x} + (1 - m)\, c^2 e^{2x} = 0 \Longrightarrow \lambda_0 = 0$$

Thus $\lambda_0 = 0$ and $f(x) = ce^x$ with $c > 0$.

VI). M is Ricci flat and $M = I \times_{f_1} F_1$ with $\dim F_1 = 1$. Then

$$\left\{ \begin{array}{l} \frac{f_1'}{f_1} - \frac{f_1''}{f_1} = 0 \\ -f_1 f_1'' + f_1 f_1' = 0 \end{array} \right. \tag{15}$$

We obtain that $f_1' = f_1''$ which implies $f_1(x) = ce^x$, with $c > 0$.

VII). M is Ricci flat and $M = I \times_{f_1} F_1 \times_{f_2} F_2$ with $\dim F_1 = \dim F_2 = 1$. Then

$$\left\{ \begin{array}{l} \frac{f_1'}{f_1} - \frac{f_1''}{f_1} + \frac{f_2'}{f_2} - \frac{f_2''}{f_2} = 0 \\ -f_1 f_1'' + \left(f_1^2 - f_1 f_1'\right) \frac{f_2'}{f_2} - f_1^2 + 2 f_1 f_1' = 0 \\ -f_2 f_2'' + \left(f_2^2 - f_2 f_2'\right) \frac{f_1'}{f_1} - f_2^2 + 2 f_2 f_2' = 0 \end{array} \right. \tag{16}$$

The system (16) is equivalent to

$$\left\{ \begin{array}{l} \frac{f_1''}{f_1} + \frac{f_2''}{f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} \\ -\frac{f_1''}{f_1} + \frac{f_2'}{f_2} - \frac{f_1' f_2'}{f_1 f_2} - 1 + 2\frac{f_1'}{f_1} = 0 \\ -\frac{f_2''}{f_2} + \frac{f_1'}{f_1} - \frac{f_1' f_2'}{f_1 f_2} - 1 + 2\frac{f_2'}{f_2} = 0 \end{array} \right. \tag{17}$$

Summing the second and the third equations and using the first equation we obtain $\left(\frac{f_1'}{f_1} - 1\right)\left(1 - \frac{f_2'}{f_2}\right) = 0$. Hence

a). $\frac{f_1'}{f_1} = 1$ and thus $f_1(x) = c_1 e^x$ and from the first equation $f_2(x) = c_2 e^x$, with $c_1, c_2 > 0$.

b). $\frac{f_2'}{f_2} = 1$ and thus $f_2(x) = c_2 e^x$ and from the first equation $f_1(x) = c_1 e^x$, with $c_1, c_2 > 0$.

VIII). M is Ricci flat and $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times ... \times_{f_m} F_m$ with $\dim F_i = 1$ for every $i \in \{1, ..., m\}$ and $m \geq 3$. Then for every $i \in \{1, ..., m\}$ the system (2) becomes

$$\left\{ \begin{array}{l} \sum\limits_{i=1}^{m} \left(\frac{f_i'}{f_i} - \frac{f_i''}{f_i} \right) = 0 \\ -f_i f_i'' + \left(f_i^2 - f_i f_i'\right) \sum\limits_{j \neq i,\, j=1}^{m} \frac{f_j'}{f_j} + (1-m) f_i^2 + m f_i f_i' = 0 \end{array} \right. \tag{18}$$

Dividing by f_i^2 the second equation of (18) one can obtain the equivalent form

$$\begin{cases} \sum_{i=1}^{m}\left(\frac{f_i'}{f_i}-\frac{f_i''}{f_i}\right)=0 \\ -\frac{f_i''}{f_i}+\left(1-\frac{f_i'}{f_i}\right)\sum_{j\neq i,\ j=1}^{m}\frac{f_j'}{f_j}+(1-m)+m\frac{f_i'}{f_i}=0 \text{ for every } i\in\{1,...,m\} \end{cases} \tag{19}$$

We make now the following notations: $h_i = \frac{f_i'}{f_i}$ for every $i \in \{1,...,m\}$ and $H = \sum_{i=1}^{m}\frac{f_i'}{f_i} = \sum_{i=1}^{m} h_i$. We remark that $\frac{f_i''}{f_i} = h_i^2 + h_i'$ for every $i \in \{1,...,m\}$. The system (19) becomes

$$\begin{cases} \sum_{i=1}^{m}\left[h_i - h_i^2 - h_i'\right]=0 \\ h_i' = h_i(m-1-H)-(m-1-H) \text{ for every } i\in\{1,...,m\} \end{cases} \tag{20}$$

Summing over i the above m equations of (20) we obtain $H' = -H^2 + (2m-1)H - m(m-1) = (m-1-H)(H-m)$.

a). If $H = m-1$, then $h_i' = 0 \Longrightarrow h_i(x) = \varepsilon_i$ for every $i \in \{1,...,m\}$. From $\frac{f_i'}{f_i} = h_i = \varepsilon_i$ we obtain $f_i(x) = \eta_i e^{\varepsilon_i x}$, with $\eta_i > 0$. Since $H = \sum_{i=1}^{m} h_i$ we obtain that $\sum_{i=1}^{m}\varepsilon_i = m-1$. The first relation implies

$$\sum_{i=1}^{m}\left[h_i - h_i^2 - h_i'\right] = 0 \Longrightarrow$$
$$\sum_{i=1}^{m}\left[\varepsilon_i - \varepsilon_i^2 - 0\right] = 0 \Longrightarrow \sum_{i=1}^{m}\varepsilon_i = \sum_{i=1}^{m}\varepsilon_i^2 = m-1$$

b). If $H = m$, then $h_i' = -h_i + 1$ implies $h_i(x) = \varepsilon_i e^{-x} + 1$, with $\varepsilon_i \in \mathbb{R}$. The first relation implies

$$\sum_{i=1}^{m}\left[h_i - h_i^2 - h_i'\right] = 0 \Longrightarrow$$
$$\left(-e^{-2x}\right)\sum_{i=1}^{m}\varepsilon_i^2 = 0 \Longrightarrow \sum_{i=1}^{m}\varepsilon_i^2 = 0$$

Hence $\varepsilon_i = 0$, $h_i(x) = 1$ and $\frac{f_i'}{f_i} = h_i = 1$ implies $f_i(x) = \eta_i e^x$, with $\eta_i > 0$ for every $i \in \{1,...,m\}$.

c). If $H \neq m-1$ and $H \neq m$, then $H' = -\left[(H - \frac{2m-1}{2})^2 - \frac{1}{4}\right]$ implies $H(x) = m - \frac{K}{e^x+K}$, where $K \in \mathbb{R}^*$ and one primitive of H is $\int H(x)dx = (m-1)x + \ln(e^x + K)$. Since $h_i' = \left(-1 + \frac{K}{e^x+K}\right)(h_i - 1)$ we have $h_i(x) = \frac{\varepsilon_i}{e^x+K} + 1$, with $\varepsilon_i \in \mathbb{R}$.

We remark that since $H = \sum_{i=1}^{m} k_i h_i$ we obtain $\sum_{i=1}^{m} \varepsilon_i = -K$. Also the relation $\sum_{i=1}^{m} \left[h_i - h_i^2 - h_i^{'} \right] = 0$ implies $\sum_{i=1}^{m} \varepsilon_i^2 = K^2$. Hence $\frac{f_i^{'}}{f_i} = h_i = 1 + \frac{\varepsilon_i}{e^x + K} \Longrightarrow f_i(x) = \eta_i e^{\left(1+\frac{\varepsilon_i}{K}\right)x} (e^x + K)^{-\frac{\varepsilon_i}{K}}$, where $K \in \mathbb{R}^*$, $\eta_i > 0$, $\varepsilon_i \in \mathbb{R}$ for every $i \in \{1, ..., m\}$ such that $\sum_{i=1}^{m} \varepsilon_i = -K$ and $\sum_{i=1}^{m} \varepsilon_i^2 = K^2$.

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