Contraction-Type Functions and Some Applications to GIIFS[1]

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Abstract

The aim of this paper is to study some properties of the contraction-type functions of one or multiple variables such as weak Meir-Keeler, Meir-Keeler, strong Meir-Keeler, contractive and non-expansive functions. We establish some equivalences on compact metric spaces. We also investigate the families of uniformly strong Meir-Keeler functions and their relation to Meir-Keeler-type functions. In the end, we give some applications to generalized infinite iterated function systems.

Keywords: *contraction, contractive function, Meir-Keeler type functions, non-expansive function.*

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1. ***Introduction***

Iterated function systems (shortly $IFSs$) were introduced in their present form by J. Hutchinson ([7]) and popularized by M. Barnsley ([1]). There is a current effort to extend Hutchinson's classical framework for fractals to more general spaces and infinite iterated function systems ($IIFSs$).

Results concerning countable iterated function systems have been obtained by N. A. Secelean for the case when the attractor is compact ([17]). R. Miculescu and A. Mihail ([12], [13]) provided a general framework where the attractors are nonempty closed and bounded subsets of complete metric spaces and where the ($IFSs$) can be infinite. K. Leśniak ([8]) presented a multivalued approach of ($IIFSs$). I. Chiţescu and R. Miculescu ([3]) presented an example of a fractal generated by Hutchinson's procedure embedded in an infinite dimensional Banach space. D. Dumitru and A. Mihail ([4]) constructed the shift space of an (IFS) consisting of ε-locally Meir-Keeler functions with $\varepsilon > 0$. A. Mihail ([14]) described the shift space associated to a generalized iterated function system ($GIFS$). Some other results and examples can be found in ([5], [6], [10], [18], [19]). Results concerning fixed point theorems can be found in ([2], [9], [15], [16], [20]). The notion of Meir-Keeler function was

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first introduced in ([11]) and the notion of φ-contraction was used in ([2]). Also, many generalizations of contractions were studied in ([16]).

The aim of this paper is to study some properties of the contraction-type functions of one or multiple variables and to give some applications of them to generalized infinite iterated function systems.

2. ***Contraction-type functions of one variable***

Let (X,d) be a metric space and $f : X \longrightarrow X$ a function. Then:

Definition 1. ([2], [9], [16]) a). f is called a *contraction* if there exists $\alpha \in [0,1)$ such that $d(f(x),f(y)) \leq \alpha \cdot d(x,y)$ for every $x,y \in X$. In this case the *Lipschitz constant* $Lip(f) = \sup\limits_{x\neq y} \frac{d(f(x),f(y))}{d(x,y)} < 1$.

b). f is called *uniformly strong Meir-Keeler* if for every $\eta > 0$ there exist $\delta, \lambda > 0$ such that $d(x,y) < \eta + \delta$ implies $d(f(x), f(y)) \leq \eta - \lambda$.

c). f is called *strong Meir-Keeler* if for every $\eta > 0$ there exists $\delta > 0$ such that $d(x,y) < \eta + \delta$ implies $d(f(x), f(y)) < \eta$.

d). f is called *contractive* if $d(f(x), f(y)) < d(x,y)$ for every $x \neq y$ from X.

e). f is called *non-expansive* if $d(f(x), f(y)) \leq d(x,y)$ for every $x, y \in X$.

Definition 2. A family of functions $f_i : X \longrightarrow X$ for every $i \in I$ is called *uniformly strong Meir-Keeler* if for every $\eta > 0$ there exist $\delta, \lambda > 0$ such that $d(x,y) < \eta + \delta$ implies $d(f_i(x), f_i(y)) \leq \eta - \lambda$ for every $i \in I$.

Remark 3. We remark that if f is uniformly strong Meir-Keeler or strong Meir-Keeler or contractive or non-expansive, then $Lip(f) \leq 1$.

Remark 4. In ([11]) A. Meir and A. Keeler use a different definition for a Meir-Keeler function: for every $\eta > 0$ there exists $\delta > 0$ such that $\eta \leq d(x,y) < \eta + \delta$ implies $d(f(x), f(y)) < \eta$. This condition, apparently "weaker", is in fact equivalent to the one we use in Definition 1, point c).

Indeed, for $\eta > 0$ let $\delta > 0$ according to the "weaker" condition. For $x \neq y$ in X with $d(x,y) < \eta + \delta$ we can have two cases:

i). $d(x,y) \geq \eta$ and thus $d(f(x), f(y)) < \eta$.

ii). $d(x,y) < \eta$. Let $d(x,y) = \eta_1 < \eta$. We can consider $\delta_1 > 0$ such that $\eta_1 \leq d(a,b) < \eta_1 + \delta_1$ implies $d(f(a), f(b)) < \eta_1$. We take $a = x$, $b = y$ and we obtain $d(f(x), f(y)) < \eta_1 < \eta$.

The following result is a well-known fixed point theorem.

Theorem 5. *([15], [21]) Let (X,d) be a complete metric space and $f : X \longrightarrow X$ a contraction or a strong Meir-Keeler function. Then f has a unique fixed point, i.e. there exists a unique $\alpha \in X$ such that $f(\alpha) = \alpha$.*

Concerning the notions introduced in Definition 1. we have the following implications.

Theorem 6. *Let (X,d) be a metric space and $f : X \longrightarrow X$ a function. Then*

(1) f contraction $\Longrightarrow$ (2) f uniformly strong Meir-Keeler $\Longrightarrow$ (3) f strong Meir-Keeler $\Longrightarrow$
(4) f contractive $\Longrightarrow$ (5) f non-expansive

Proof:

(1) $\Longrightarrow$ (2)

For every $\eta > 0$ there exist $\delta = \frac{\eta(1-c)}{2c} > 0$ and $\lambda = \frac{\eta(1-c)}{2} > 0$. Let $x, y \in X$ such that $d(x,y) < \eta + \delta$. Then

$$d\left(f\left(x\right), f\left(y\right)\right) \leq c \cdot d\left(x, y\right) < c(\eta + \delta) = c\left[\eta + \frac{\eta(1-c)}{2c}\right] = \frac{\eta(1+c)}{2} = \eta - \lambda.$$

(2) $\Longrightarrow$ (3)

For every $\eta > 0$ there exist $\delta, \lambda > 0$ such that $d\left(x, y\right) < \eta + \delta$ implies $d\left(f\left(x\right), f\left(y\right)\right) \leq \eta - \lambda < \eta$ and thus f is strong Meir-Keeler.

(3) $\Longrightarrow$ (4)

Let $x \neq y$ and $\eta = d\left(x, y\right) > 0$. Then there exists $\delta > 0$ such that $d(x,y) < d(x,y) + \delta = \eta + \delta$ implies $d\left(f\left(x\right), f\left(y\right)\right) < \eta = d\left(x, y\right)$ and thus f is contractive.

(4) $\Longrightarrow$ (5)

We have that $d\left(f\left(x\right), f\left(y\right)\right) < d\left(x, y\right)$ for every $x \neq y$ which implies $d\left(f\left(x\right), f\left(y\right)\right) \leq d\left(x, y\right)$ for every $x, y \in X$.

The following result generalizes the implication (1) $\Longrightarrow$ (2) from theorem 6.

Theorem 7. *Let (X, d) be a metric space and $f_i : X \longrightarrow X$ a contraction for every $i \in I$ with $c = \sup\limits_{i \in I} Lip(f_i) < 1$. Then the family $(f_i)_{i \in I}$ is uniformly strong Meir-Keeler.*

Proof: Let $\eta > 0$ be fixed and $x, y \in X$ such that $d(x,y) < \eta + \delta$. Then $d\left(f_i\left(x\right), f_i\left(y\right)\right) \leq Lip\left(f_i\right) \cdot d(x,y) \leq c \cdot d(x,y) < c(\eta + \delta)$. Furthermore, $c(\eta + \delta) \leq \eta - \lambda \Longleftrightarrow \lambda \leq \eta(1-c) - c\delta$. We remark that $\eta(1-c) - c\delta > 0 \Longleftrightarrow \delta < \frac{\eta(1-c)}{c}$. Hence the family $(f_i)_{i \in I}$ is uniformly strong Meir-Keeler.

Theorem 8. *Let (X, d) be a compact metric space and $f : X \longrightarrow X$ a contractive function. Then f is strong Meir-Keeler.*

Proof: Let $\eta > 0$. We consider the set $A_\eta = \{(x, y) \in X \times X \mid d\left(x, y\right) \geq \eta\}$. Then $A_\eta \subset X \times X$ is a closed set included in a compact space, thus A_η is also compact. Let $\varphi : X \times X \longrightarrow \mathbb{R}_+$ defined by $\varphi\left(x, y\right) = d\left(x, y\right) - d\left(f\left(x\right), f\left(y\right)\right)$.Then φ is a continuous function. Thus there exists $(a, b) \in A_\eta$ such that $\inf\limits_{(x,y) \in A_\eta} \varphi\left(x, y\right) = \varphi\left(a, b\right) = d\left(a, b\right) - d\left(f\left(a\right), f\left(b\right)\right) > 0$. Let $\delta_1 = \inf\limits_{(x,y) \in A_\eta} \varphi\left(x, y\right) > 0$ and $\delta \in \left(0, \delta_1\right)$. Then for every $(x, y) \in A_\eta$ we have $\varphi\left(x, y\right) = d\left(x, y\right) - d\left(f\left(x\right), f\left(y\right)\right) \geq \delta_1 > \delta \Longrightarrow d\left(f\left(x\right), f\left(y\right)\right) < d\left(x, y\right) - \delta$. Now let $(x, y) \in A_\eta$ such that $d(x,y) < \eta + \delta$. Hence $d\left(f\left(x\right), f\left(y\right)\right) < d\left(x, y\right) - \delta < \eta + \delta - \delta = \eta$ and thus f is strong Meir-Keeler.

Theorem 9. *Let (X, d) be a compact metric space and $f_i : X \longrightarrow X$ for every $i \in \{1, ..., n\}$, $n \geq 1$. If f_i is strong Meir-Keeler for every $i \in \{1, ..., n\}$, then the family $(f_i)_{i=\overline{1,n}}$ is uniformly strong Meir-Keeler.*

Proof: a). Let $n = 1$ and $f = f_1$ a strong Meir-Keeler function. Then for every $\eta > 0$ there exists $\delta > 0$ such that $d(x,y) < \eta + \delta$ implies $d(f(x), f(y)) < \eta$. But if we consider $\delta' \in (0,\delta)$ then $d(x,y) \le \eta + \delta' < \eta + \delta$ implies $d(f(x), f(y)) < \eta$. In this way we can rewrite the strong Meir-Keeler condition as following: for every $\eta > 0$ there exists $\delta > 0$ such that $d(x,y) \le \eta + \delta$ implies $d(f(x), f(y)) < \eta$. Let now $A(\eta) = \{(x,y) \in X \times X \mid d(x,y) \le \eta + \delta\}$. Since $A(\eta)$ is a closed set included in a compact space, it follows that $A(\eta)$ is also compact.

Hence $\sup\limits_{(x,y)\in A(\eta)} d(f(x), f(y)) = d(f(r), f(s)) = \eta' < \eta$ with $(r,s) \in A(\eta)$. Let $\lambda = \eta - \eta' > 0$. Then for every $(x,y) \in A(\eta)$ we have $d(f(x), f(y)) \le \eta' = \eta - \lambda$. Thus the family $(f) = (f_1)$ is uniformly strong Meir-Keeler.

b). Let $n \ge 2$, $\eta > 0$ be fixed and $f_1, ..., f_n : X \longrightarrow X$ be strong Meir-Keeler functions. From point a) there exist δ_i, $\lambda_i > 0$ such that $d(x,y) < \eta + \delta_i$ implies $d(f_i(x), f_i(y)) \le \eta - \lambda_i$ for every $i \in \{1, ..., n\}$. We consider now $\delta = \min\limits_{i=\overline{1,n}} \delta_i > 0$, $\lambda = \min\limits_{i=\overline{1,n}} \lambda_i > 0$ and thus $d(x,y) < \eta + \delta$ implies $d(f_i(x), f_i(y)) \le \eta - \lambda$ for every $i \in \{1, ..., n\}$. Thus the family $(f_i)_{i=\overline{1,n}}$ is uniformly strong Meir-Keeler.

Remark 10. We remark that on compact spaces f uniformly strong Meir-Keeler $\Longleftrightarrow$ f strong Meir-Keeler $\Longleftrightarrow$ f contractive. We also deduce the well-known result that on compact spaces a contractive function has a unique fixed point. It also follows that on compact spaces an uniformly strong Meir-Keeler function has a unique fixed point.

Theorem 5 is not valid for contractive functions.

Example 11. The function $f : [0,\infty) \longrightarrow [0,\infty)$, $f(x) = \frac{(x+1)^2}{x+2}$ is contractive and has no fixed points. Indeed, we have that $f'(x) = \frac{x^2+4x+3}{x^2+4x+4}$ and thus $0 < f'(x) < 1$ for every $x \ge 0$. Hence, according to Langrange's theorem, f is contractive. Also, since $f(x) > x$ for every $x \ge 0$, it follows that f has no fixed points.

Concerning Theorem 6, the conversed implications are not true.

Example 12. Let $f : [0,\infty) \longrightarrow [0,\infty)$, $f(x) = x$ (the identity function). Then clearly f is non-expansive but not contractive.

Example 13. Let $f : [0,\infty) \longrightarrow [0,\infty)$, $f(x) = \frac{(x+1)^2}{x+2}$. Then f is contractive but not strong Meir-Keeler. Indeed, we have that f is a contraction with no fixed points from Example 11. According to Theorem 5, it follows that f is not strong Meir-Keeler.

Example 14. Let $X = [0,1] \cup \{3, 4, 6, 7, ..., 3n, 3n+1, ...\}$, $n \ge 1$ endowed with the Euclidean distance and $f : X \longrightarrow X$ be defined by

$$f(x) = \begin{cases} \frac{x}{2} & , x \in [0,1] \\ 0 & , x = 3n,\ n \ge 1 \\ 1 - \frac{1}{n+2} & , x = 3n+1,\ n \ge 1 \end{cases}$$

It is proven in ([11]) that f is strong Meir-Keeler. We remark that f is not uniformly strong Meir-Keeler.

Indeed, let $\eta = 1$ and suppose that there exist $\delta, \lambda > 0$ such that $|x - y| < 1+\delta$ implies $|f(x)-f(y)| \leq 1-\lambda$. Consider $x_n = 3n$ and $y_n = 3n+1$ for every $n \geq 1$. Then $|x_n - y_n| = 1 < 1 + \delta$ and $|f(x_n) - f(y_n)| = \left|0 - \left(1 - \frac{1}{n+2}\right)\right| = 1 - \frac{1}{n+2} \leq 1 - \lambda$ for every $n \geq 1$, which is a contradiction. Thus f is not uniformly strong Meir-Keeler.

Example 15. Let $f : [0, 1] \longrightarrow [0, 1]$, $f(x) = x - x^2$. Then f is uniformly strong Meir-Keeler but is not a contraction.

Indeed, f is not a contraction since $f^{'}(x) = 1 - 2x$ implies $Lip(f) = \sup_{x\in[0,1]} f^{'}(x) = 1$. Also, f is contractive since for $x \neq y$ from the interval $[0, 1]$ we have:

$$\begin{gathered}
|f(x) - f(y)| < |x - y| \Longleftrightarrow \\
|x - y| \cdot |1 - x - y| < |x - y| \Longleftrightarrow \\
|1 - x - y| < 1.
\end{gathered}$$

Indeed, $-1 \leq 1 - x - y \leq 1$ implies $|1 - x - y| \leq 1$ with equality if and only if $x = y = 0$ or $x = y = 1$, which is in contradiction with $x \neq y$. Hence, using Theorem 8 f is strong Meir-Keeler. Also, using Theorem 9 f is uniformly strong Meir-Keeler.

Theorem 9 is not valid for an infinite family of strong Meir-Keeler functions.

Example 16. Let $f_n : [0, 1] \longrightarrow [0, 1]$ be defined by $f_n(x) = \left(1 - \frac{1}{n}\right) x$ for every $x \in [0, 1]$ and $n \geq 2$. Then f_n is a strong Meir-Keeler function for every $n \geq 2$ but the family $(f_n)_{n\geq2}$ is not uniformly strong Meir-Keeler.

Indeed, we have that $Lip(f_n) = 1 - \frac{1}{n} < 1$ and thus f_n is a contration for every $n \geq 2$. According to theorem 6 it follows that f_n is a strong Meir-Keeler function for every $n \geq 2$. Take now $\eta = 1$, $\delta, \lambda > 0$, $x = 1, y = 2$ and $n_0 \geq 2$ such that $1 - \frac{1}{n_0} > 1 - \lambda$.

Then we obtain that $|x-y| = 1 < 1+\delta$ but $|f_{n_0}(1)-f_{n_o}(0)| = \left|1 - \frac{1}{n_0}\right| = 1 - \frac{1}{n_0} > 1 - \lambda$ and therefore the family $(f_n)_{n\geq2}$ is not uniformly strong Meir-Keeler.

Open problem 1. The author thinks that the notion of uniformly strong Meir-Keeler is equivalent to the notion of φ-contraction, which is $d(f(x), f(y)) \leq \varphi(d(x, y))$, where $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is non-decreasing, right continuous and $\varphi(x) < x$ for every $x \in (0,\infty)$.

3. ***Contraction-type functions of multiple variables***

Let (X,d) be a metric space and $m \geq 1$. The space X^m will be endowed with the metric $\widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) = \max\limits_{i=\overline{1,m}} d(x_i,y_i)$. We consider the following contraction-type definitions:

Definition 17. a). A function $f : X^m \longrightarrow X$ is called a *contraction* if

$$Lip(f) = \sup\left\{\frac{d(f(x_1,...,x_m),f(y_1,...,y_m))}{\max\{d(x_1,y_1),...,d(x_m,y_m)\}} \mid x_i,y_i \in X,\ \ \max d(x_i,y_i) > 0\right\} < 1$$

b). A function $f : X^m \longrightarrow X$ is called *weak Meir-Keeler* if for every $\eta > 0$ there exists $\delta > 0$ such that $\eta \leq d(x_i,y_i) < \eta + \delta$ for every $i \in \{1,...,m\}$ implies $d(f(x_1,...,x_m), f(y_1,...,y_m)) < \eta$.

c). A function $f : X^m \longrightarrow X$ is called *Meir-Keeler* if for every $\eta_i > 0$ there exists $\delta > 0$ such that $\eta_i \leq d(x_i,y_i) < \eta_i + \delta$ for every $i \in \{1,...,m\}$ implies $d(f(x_1,...,x_m), f(y_1,...,y_m)) < \max\limits_{i=\overline{1,m}} \eta_i$.

d). A function $f : X^m \longrightarrow X$ is called *strong Meir-Keeler* if for every $\eta > 0$ there exists $\delta > 0$ such that $d(x_i,y_i) < \eta + \delta$ for every $i \in \{1,...,m\}$ implies $d(f(x_1,...,x_m), f(y_1,...,y_m)) < \eta$.

e). A family of functions $f_i : X^m \longrightarrow X$ for every $i \in I$ is said to be *uniformly strong Meir-Keeler* if for every $\eta > 0$ there exist δ, $\lambda > 0$ such that $d(x_i,y_i) < \eta + \delta$ for every $i \in \{1,...,m\}$ implies $d(f_i(x_1,...,x_m), f_i(y_1,...,y_m)) \leq \eta - \lambda$ for every $i \in I$.

f). A function $f : X^m \longrightarrow X$ is called *contractive* if $d(f(x_1,...,x_m), f(y_1,...,y_m)) < \max\{d(x_1,y_1),...,d(x_m,y_m)\}$ for every $(x_1,...,x_m) \neq (y_1,...,y_m)$ from X^m.

g). A function $f : X^m \longrightarrow X$ is called *non-expansive* if $d(f(x_1,...,x_m), f(y_1,...,y_m)) \leq \max\{d(x_1,y_1),...,d(x_m,y_m)\}$ for every $(x_1,...,x_m)$, $(y_1,...,y_m) \in X^m$.

Remark 18. We remark that if f is strong Meir-Keeler or contractive or non-expansive, then $Lip(f) \leq 1$. We also remark that Meir-Keeler implies weak Meir-Keeler and strong Meir-Keeler implies weak Meir-Keeler.

Theorem 19. *Let (X,d) be a complete metric space and $f : X^m \longrightarrow X$ a contraction or a strong Meir-Keeler function. Then f has a unique fixed point, i.e. there exists a unique $\alpha \in X$ such that $f(\alpha,\alpha,...,\alpha) = \alpha$.*

Proof: a). f is a contraction.

i). If $m = 1$, we have the well-known Banach's contraction theorem.

ii). If $m \geq 2$, we define $g : X \longrightarrow X$ by $g(x) = f(x,...,x)$ for every $x \in X$. Since $f : X^m \longrightarrow X$ is a contraction, then

$$Lip(f) = \sup\left\{\frac{d(f(x_1,...,x_m), f(y_1,...,y_m))}{\max\{d(x_1,y_1),...,d(x_m,y_m)\}} \mid x_i,y_i \in X, \max d(x_i,y_i) > 0\right\} < 1.$$

We take $x_1 = x_2 = ... = x_m = x$ and $y_1 = y_2 = ... = y_m = y$ and thus, in particular $\sup\left\{\frac{d(f(x,...,x),f(y,...,y))}{d(x,y)} \mid x, y \in X, \ \max d(x,y) > 0\right\} < 1$ which implies $\sup\limits_{x\neq y}\frac{d(g(x),g(y))}{d(x,y)} < 1$. That means g is a contraction and thus from point i) g has a unique $\alpha \in X$ such that $g(\alpha) = \alpha \Longleftrightarrow f(\alpha, ..., \alpha) = \alpha$.

b). f is strong Meir-Keeler.

i). If $m = 1$, we obtain a very well–known result which is proved in [21].

ii). If $m \geq 2$, we define the function $g : X \to X$, given by $g(x) = f(x, ..., x)$ for every $x \in X$. Since $f : X^m \to X$ is strong Meir-Keeler then for every $\eta > 0$ there exists $\delta > 0$ such that $x_i, y_i \in X$ with $d(x_i, y_i) < \eta + \delta$ for every $i \in \{1, ..., m\}$ implies $d(f(x_1, ..., x_m), f(y_1, ..., y_m)) < \eta$. We can take $x_1 = x_2 = ... = x_m = x$ and $y_1 = y_2 = ... = y_m = y$ with $x, y \in X$, $d(x, y) < \eta + \delta$ and we obtain that $d(f(x, ..., x), f(y, ..., y)) < \eta$. This means that for every $x, y \in X$ with $d(x, y) < \eta + \delta$ we have $d(g(x), g(y)) < \eta$. Thus g is strong Meir-Keeler and from i) it follows that there exists a unique $\alpha \in X$ such that $g(\alpha) = \alpha$ which implies $f(\alpha, ..., \alpha) = \alpha$.

Theorem 20. *Let (X, d) be a metric space, $m \geq 1$ and $f : X^m \longrightarrow X$ a function. Then*

(1) f contraction $\Longrightarrow$ (2) f uniformly strong Meir-Keeler $\Longrightarrow$ (3) f strong Meir-Keeler $\Longrightarrow$
(4) f contractive $\Longrightarrow$ (5) f non-expansive.

Proof: (1) $\Longrightarrow$ (2). We have

$$Lip\,(f) = \sup\left\{\frac{d(f(x_1,...,x_m),f(y_1,...,y_m))}{\max\{d(x_1,y_1),...,d(x_m,y_m)\}} \mid x_i,y_i \in X, \ \max d\,(x_i, y_i) > 0\right\} < 1 \Longrightarrow$$
$$\frac{d(f(x_1,...,x_m),f(y_1,...,y_m))}{\max\{d(x_1,y_1),...,d(x_m,y_m)\}} \leq Lip(f) < 1 \Longrightarrow$$
$$d\,(f\,(x_1, ..., x_m)\,, f\,(y_1, ..., y_m)) \leq Lip(f) \cdot \max\{d\,(x_1, y_1)\,, ..., d\,(x_m, y_m)\}$$

for every $\max\{d\,(x_1, y_1)\,, ..., d\,(x_m, y_m)\} > 0$.

a). If $\max\{d(x_1, y_1), ..., d(x_m, y_m)\} = 0$, then $d(x_i, y_i) = 0 < \eta+\delta$ implies $d(f(x_1, ..., x_m), f(y_1, ..., y_m)) = 0 < \eta - \lambda$.

b). If $\max\{d\,(x_1, y_1)\,, ..., d\,(x_m, y_m)\} > 0$, then $d(x_i, y_i) < \eta + \delta$ implies $d(f(x_1, ..., x_m), f(y_1, ..., y_m)) \leq Lip(f) \max\{d(x_1, y_1), ..., d(x_m, y_m)\} < Lip(f)$ $(\eta+\delta)$. But, $Lip(f)(\eta+\delta) \leq \eta-\lambda \Longleftrightarrow \lambda \leq \eta-Lip(f)(\eta+\delta) = \eta(1-Lip(f))-$ $Lip(f)\delta$. We also remark that $\eta(1 - Lip(f)) - Lip(f)\delta > 0 \Longleftrightarrow \delta < \frac{\eta(1-Lip(f))}{Lip(f)}$ which means that f is strong Meir-Keeler.

(2) $\Longrightarrow$ (3) follows from Definition 17.

(3) $\Longrightarrow$ (4) Let $(x_1, ..., x_m) \neq (y_1, ..., y_m)$ and $\eta = \max\{d(x_1, y_1), ..., d(x_m, y_m)\} > 0$. Then there exists $\delta > 0$ such that $d(x_i, y_i) < \eta + \delta$ implies $d\,(f\,(x_1, ..., x_m)\,, f\,(y_1, ..., y_m)) < \eta = \max\{d\,(x_1, y_1)\,, ..., d\,(x_m, y_m)\}$ and thus f is contractive.

(4) $\Longrightarrow$ (5) From $d(f(x_1, ..., x_m), f(y_1, ..., y_m)) < \max\{d(x_1, y_1), ..., d(x_m, y_m)\}$ for every $(x_1, ..., x_m) \neq (y_1, ..., y_m)$ we obtain $d(f(x_1, ..., x_m), f(y_1,$

$...,y_m)) \leq \max\{d(x_1,y_1),...,d(x_m,y_m)\}$ for every $(x_1,...,x_m),(y_1,...,y_m) \in X^m$.

Remark 21. Similar to Theorem 20 we can also prove the following implications: f contraction $\Longrightarrow f$ Meir-Keeler $\Longrightarrow f$ weak Meir-Keeler and f contraction $\Longrightarrow f$ uniformly strong Meir-Keeler $\Longrightarrow f$ strong Meir-Keeler $\Longrightarrow$ f weak Meir-Keeler.

Theorem 22. *Let (X,d) be a compact metric space, $m \geq 1$ and $f : X^m \longrightarrow X$ a function. Then:*
a). If f is contractive, then f is strong Meir-Keeler.
b). If f is strong Meir-Keeler, then f is uniformly strong Meir-Keeler.

Proof: a). Let $\eta > 0$. We consider the set:

$$A_\eta = \left\{((x_1,...,x_m),(y_1,...,y_m)) \in X^m \times X^m \mid \widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) \geq \eta\right\}.$$

Then $A_\eta \subset X^m \times X^m$ is a closed set included in a compact space, thus A_η is also compact. Let $\varphi : X^m \times X^m \longrightarrow \mathbb{R}_+$ be defined by:

$$\varphi((x_1,...,x_m),(y_1,...,y_m)) = \widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) - d(f(x_1,...,x_m),f(y_1,...,y_m))$$

Then φ is a continuous function. Thus there exists $((a_1,...,a_m),(b_1,...,b_m)) \in A_\eta$ such that $\inf_{A_\eta}\varphi((x_1,...,x_m),(y_1,...,y_m)) = \varphi((a_1,...,a_m),(b_1,...,b_m)) = \widetilde{d}((a_1,...,a_m),(b_1,...,b_m)) - d(f(a_1,...,a_m),f(b_1,...,b_m)) > 0$. Let $\delta_1 = \inf_{A_\eta}\varphi((x_1,...,x_m),(y_1,...,y_m)) > 0$ and $\delta \in (0,\delta_1)$. Then for every $((x_1,...,x_m),(y_1,...,y_m)) \in A_\eta$ we have:

$$\varphi((x_1,...,x_m),(y_1,...,y_m)) = \widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) - d(f(x_1,...,x_m),f(y_1,...,y_m)) \geq \delta_1 > \delta \Longrightarrow d(f(x_1,...,x_m),f(y_1,...,y_m)) < \widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) - \delta$$

Now let $((x_1,...,x_m),(y_1,...,y_m)) \in A_\eta$ such that $\widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) < \eta+\delta \Longleftrightarrow \max_{i=\overline{1,m}} d(x_i,y_i) < \eta+\delta \Longleftrightarrow d(x_i,y_i) < \eta+\delta$ for every $i \in \{1,...,m\}$. Hence $d(f(x_1,...,x_m),f(y_1,...,y_m)) < \widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) - \delta < \eta+\delta-\delta = \eta$ and thus f is strong Meir-Keeler.

b). Let f be a strong Meir-Keeler function. Then for every $\eta > 0$ there exists $\delta > 0$ such that $\widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) < \eta+\delta \Longleftrightarrow \max_{i=\overline{1,m}} d(x_i,y_i) < \eta+\delta \Longleftrightarrow d(x_i,y_i) < \eta+\delta$ for every $i \in \{1,...,m\}$ implies $d(f(x_1,...,x_m),f(y_1,...,y_m)) < \eta$. But if we consider $\delta' \in (0,\delta)$ then $\widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) \leq$

$\eta + \delta' < \eta + \delta$ implies $d(f(x_1,...,x_m), f(y_1,...,y_m)) < \eta$. In this way we can rewrite the strong Meir-Keeler condition as following: for every $\eta > 0$ there exists $\delta > 0$ such that $\widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) \leq \eta + \delta$ implies $d(f(x_1,...,x_m), f(y_1,...,y_m)) < \eta$. Let now

$$A(\eta) = \left\{((x_1,...,x_m),(y_1,...,y_m)) \in X^m \times X^m \mid \widetilde{d}((x_1,...,x_m),(y_1,...,y_m)) \leq \eta + \delta\right\}$$

Since $A(\eta)$ is a closed set included in a compact space, it follows that $A(\eta)$ is also compact. Hence $\sup_{A(\eta)} d(f(x_1,...,x_m), f(y_1,...,y_m)) = d(f(r_1,...,r_m), f(s_1,...,s_m)) = \eta' < \eta$ with $((r_1,...,r_m),(s_1,...,s_m)) \in A(\eta)$. Let $\lambda = \eta - \eta' > 0$. Then for every $((x_1,...,x_m),(y_1,...,y_m)) \in A(\eta)$ we have $d(f(x_1,...,x_m), f(y_1,...,y_m)) \leq \eta' = \eta - \lambda$. Thus the function f is uniformly strong Meir-Keeler.

Remark 23. Similar to the functions of one variable, on compact spaces, we have the following equivalence for functions $f : X^m \longrightarrow X$, $m \leq 1$: f uniformly strong Meir-Keeler $\Longleftrightarrow$ f strong Meir-Keeler $\Longleftrightarrow$ f contractive.

4. *Applications to generalized infinte iterated function systems*

For a nonempty set X, we will denote by $\mathcal{P}(X)$ the set of all nonempty subsets of X, by $\mathcal{K}(X)$ the set of all nonempty compact subsets of X and by $\mathcal{B}(X)$ the set of all nonempty closed and bounded subsets of X.

Definition 24. Let (X, d) be a metric space. The *generalized Hausdorff-Pompeiu semidistance* is the application $h : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow [0, +\infty]$ defined by $h(A, B) = \max\{d(A, B),\ d(B, A)\}$, where $d(A, B) = \sup_{x\in A} d(x, B) = \sup_{x\in A}\left(\inf_{y\in B} d(x, y)\right)$. When we consider $h : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow [0, +\infty)$ or $h : \mathcal{B}(X) \times \mathcal{B}(X) \longrightarrow [0, +\infty)$, then h becomes a distance. It is well-known that when (X, d) is complete, then $(\mathcal{K}(X), h)$ and $(\mathcal{B}(X), h)$ are also complete. Further by $\mathcal{K}(X)$ and $\mathcal{B}(X)$ we will refer to $(\mathcal{K}(X), h)$ and $(\mathcal{B}(X), h)$.

Proposition 25. *Let (X, d) be a metric space and $m \geq 1$.*

1). If H and K are two nonempty subsets of X then $h(H, K) = h(\overline{H}, \overline{K})$,

2). If $(H_i)_{i\in I}$ and $(K_i)_{i\in I}$ are two families of nonempty subsets of X then

$$h\left(\bigcup_{i\in I} H_i, \bigcup_{i\in I} K_i\right) = h\left(\overline{\bigcup_{i\in I} H_i}, \overline{\bigcup_{i\in I} K_i}\right) \leq \sup_{i\in I} h(H_i, K_i).$$

3). If H and K are two nonempty subsets of X and $f : X \longrightarrow X$ is a Lipschitz function then $h(f(H), f(K)) \leq Lip(f) \cdot h(H, K)$.

4). *If* A_i *and* B_i *are nonempty subsets of* X *for every* $i \in \{1,...,m\}$ *and* $f : X^m \longrightarrow X$ *is a Lipschitz function then*

$$h(f(A_1,...,A_m), f(B_1,...,B_m)) \leq Lip(f) \cdot \max\{h(A_1,B_1),...,h(A_m,B_m)\}.$$

Proof: The assertions 1), 2) and 3) are well-known and can be found in ([1]).

4). We have

$$h(f(A_1,...,A_m), f(B_1,...,B_m)) = \max\{d(f(A_1,...,A_m), f(B_1,...,B_m)), d(f(B_1,...,B_m), f(A_1,...,A_m))\},$$

where $d(f(A_1,...,A_m), f(B_1,...,B_m)) = \sup\limits_{a_i \in A_i}\left(\inf\limits_{b_j \in B_j} d(f(a_1,...,a_m), f(b_1,...,b_m))\right)$.

But $d(f(a_1,...,a_m), f(b_1,...,b_m)) \leq Lip(f) \cdot \max\{d(a_1,b_1),...,d(a_m,b_m)\}$ for every $a_i \in A_i$ and $b_j \in B_j$.

Thus:

$$\inf_{b_j \in B_j} d(f(a_1,...,a_m), f(b_1,...,b_m)) \leq \inf_{b_j \in B_j} Lip(f) \cdot \max\{d(a_1,b_1),...,d(a_m,b_m)\} \Longrightarrow$$
$$\sup_{a_i \in A_i}\left(\inf_{b_j \in B_j} d(f(a_1,...,a_m), f(b_1,...,b_m))\right) \leq \sup_{a_i \in A_i}\left(\inf_{b_j \in B_j} Lip(f)\max\{d(a_1,b_1),...,d(a_m,b_m)\}\right) \Longrightarrow$$
$$d(f(A_1,...,A_m), f(B_1,...,B_m)) \leq Lip(f) \cdot \max\left\{\sup_{a_1 \in A_1}\left(\inf_{b_1 \in B_1} d(a_1,b_1)\right),..., \sup_{a_m \in A_m}\left(\inf_{b_m \in B_m} d(a_m,b_m)\right)\right\} \Longrightarrow$$
$$d(f(A_1,...,A_m), f(B_1,...,B_m)) \leq Lip(f) \cdot \max\{h(A_1,B_1),...,h(A_m,B_m)\}.$$

In a similar way, we obtain $d(f(B_1,...,B_m), f(A_1,...,A_m)) \leq Lip(f) \cdot \max\{h(B_1,A_1),...,h(B_m,A_m)\} = Lip(f) \cdot \max\{h(A_1,B_1),...,h(A_m,B_m)\}$ and thus the conclusion.

Definition 26. ([12]) Let (X,d) be a complete metric space. Then:

a). A family $(f_i)_{i \in I}$ of continuous functions, where $f_i : X^m \longrightarrow X$ for every $i \in I$, $m \in \mathbb{N}^*$, is said to be *bounded* if for every bounded sets $B_1, B_2, ..., B_m \subset X$, $\bigcup\limits_{i \in I} f_i(B_1, B_2, ..., B_m) \subset X$ is bounded.

b). An *infinite iterated function system* $(IIFS)$ consists of a bounded family $(f_i)_{i \in I}$ of continuous functions with $f_i : X \longrightarrow X$ for every $i \in I$ and it is denoted by $\mathcal{S} = (X, (f_i)_{i \in I})$.

c). A *generalized infinite iterated function system* $(GIIFS)$ consists of a bounded family $(f_i)_{i \in I}$ of continuous functions with $f_i : X^m \longrightarrow X$ for every $i \in I$, where $m \in \mathbb{N}^*$ and it is denoted by $\mathcal{S}^m = (X^m, (f_i)_{i \in I})$.

We remark that every $(IIFS)$ is a particular case of a $(GIIFS)$ for $m = 1$.

Definition 27. ([12]) Let (X,d) be a complete metric space and $\mathcal{S}^m = (X^m,(f_i)_{i\in I})$ a $(GIIFS)$, where $m \in \mathbb{N}^*$. The function $F_{\mathcal{S}^m} : \mathcal{B}(X)^m \longrightarrow \mathcal{B}(X)$ defined by $F_{\mathcal{S}^m}(B_1,...,B_m) = \overline{\bigcup_{i\in I} f_i(B_1,...,B_m)}$ for every $B_1,...,B_m \in \mathcal{B}(X)$ is called the *fractal operator* associated to the $(GIIFS)$ $\mathcal{S}^m$. For $m=1$ we find $F_{\mathcal{S}^1}(B) = F_{\mathcal{S}}(B) = \overline{\bigcup_{i\in I} f_i(B)}$ for every $B \in \mathcal{B}(X)$, the *fractal operator* of an infinite iterated function system.

Remark 28. We remark that the function $F_{\mathcal{S}^m}$ from above is well-defined since in the definition of a $(GIIFS)$ we suppose that the family of functions $(f_i)_{i\in I}$ is bounded. If this is not true, then the fractal operator could not be well-defined as one can see from the following counterexample: take $X=\mathbb{R}$, $I=\mathbb{N}$, $m=1$ and $f_i(x) = x+i$ for every $i \in \{1,2,...\}$. Then $F_{\mathcal{S}^1}([0,1]) = \overline{\bigcup_{i\in\mathbb{N}} f_i([0,1])} = \overline{\bigcup_{i\in\mathbb{N}} [i,i+1]} = [0,\infty) \notin \mathcal{B}(X)$.

Lemma 29. *([15]) Let $A,B \in \mathcal{B}(X)$. Then for each $\gamma > 0$* and $a \in A$ *there exists $b \in B$ such that $d(a,b) \le h(A,B) + \gamma$.*

The following theorem gives a characterization of the fractal operator associated to a $(GIIFS)$.

Theorem 30. *Let (X,d) be a metric space, $\mathcal{S}^m = (X^m,(f_i)_{i\in I})$ a $(GIIFS)$ with $m \in \mathbb{N}^*$ and the fractal operator $F_{\mathcal{S}^m} : \mathcal{B}(X)^m \longrightarrow \mathcal{B}(X)$ defined by $F_{\mathcal{S}^m}(B_1,B_2,...,B_m) = \overline{\bigcup_{i\in I} f_i(B_1,B_2,...,B_m)}$ for every $B_1,...,B_m \in \mathcal{B}(X)$. Then:*

a). If f_i is a contraction for every $i \in I$ and $c = \sup_{i\in I} Lip(f_i) < 1$, then $F_{\mathcal{S}^m}$ is a contraction with $Lip(F_{\mathcal{S}^m}) \le c < 1$.

b). If the family of functions $(f_i)_{i\in I}$ is uniformly strong Meir-Keeler, then $F_{\mathcal{S}^m}$ is strong Meir-Keeler (in particular, $F_{\mathcal{S}^m}$ is weak Meir-Keeler).

c). If f_i is contractive (or non-expansive) for every $i \in I$, then $F_{\mathcal{S}^m}$ is non-expansive.

Proof: a). Let $A_i, B_i \in \mathcal{B}(X)$ for every $i \in \{1,...,m\}$. Then:

$$\begin{gathered} h\left(F_{\mathcal{S}^m}(A_1,...,A_m), F_{\mathcal{S}^m}(B_1,...,B_m)\right) = \\ h\left(\overline{\bigcup_{i\in I} f_i(A_1,...,A_m)}, \overline{\bigcup_{i\in I} f_i(B_1,...,B_m)}\right) = \\ h\left(\bigcup_{i\in I} f_i(A_1,...,A_m), \bigcup_{i\in I} f_i(B_1,...,B_m)\right) \le \\ \sup_{i\in I} h(f_i(A_1,...,A_m), f_i(B_1,...,B_m)) \le \\ \sup_{i\in I} Lip(f_i)\cdot \max_{j=\overline{1,m}} h(A_j,B_j) = c\cdot \max_{j=\overline{1,m}} h(A_j,B_j). \end{gathered}$$

So $F_{\mathcal{S}^m}$ is a contraction with $Lip(F_{\mathcal{S}}) \le c = \sup_{i\in I} Lip(f_i) < 1$.

b). Let $\eta > 0$. Then there exist δ, $\lambda > 0$ such that $d(x_i,y_i) < \eta + \delta$ for every $i \in \{1,...,m\}$ implies $d(f_j(x_1,...,x_m), f_j(y_1,...,y_m)) \le \eta - \lambda$ for every $j \in I$. Let now $A_i, B_i \in \mathcal{B}(X)$ such that $h(A_i,B_i) < \eta + \delta$ for every $i \in \{1,...,m\}$. Then, there exists $\gamma_i > 0$ such that $h(A_i,B_i) + \gamma_i < \eta + \delta$

for every $i \in \{1,...,m\}$. Consider $\gamma = \min_{i=\overline{1,m}} \gamma_i > 0$. Thus $h(A_i, B_i) + \gamma < h(A_i, B_i) + \gamma_i < \eta + \delta$ for every $i \in \{1,...,m\}$.

To this end let $z \in \bigcup_{i \in I} f_i(A_1,...,A_m)$ be arbitrarily chosen. Then there exist $i_0 \in \{1,...,n\}$ and $x_1 \in A_1,...,x_m \in A_m$ such that $z = f_{i_0}(x_1,...,x_m)$. From lemma 29 there exist $y_1 \in B_1,...,y_m \in B_m$ such that $d(x_i, y_i) \leq h(A_i, B_i) + \gamma < \eta + \delta$ for every $i \in \{1,...,m\}$. Since $d(x_i, y_i) < \eta + \delta$ for every $i \in \{1,...,m\}$ we obtain that $d(f_{i_0}(x_1,...,x_m), f_{i_0}(y_1,...,y_m)) \leq \eta - \lambda$. Thus:

$$\inf_{w_1 \in B_1,...,w_m \in B_m} d(f_{i_0}(x_1,...,x_m), f_{i_0}(w_1,...,w_m)) = d(z, f_{i_0}(B_1,...,B_m)) \leq \eta - \lambda.$$

Hence $d\left(z, \bigcup_{i \in I} f_i(B_1,...,B_m)\right) \leq d(z, f_{i_0}(B_1,...,B_m)) \leq \eta - \lambda$. Because z was arbitrarily chosen in $\bigcup_{i \in I} f_i(A_1,...,A_m)$, we have that $d\left(\bigcup_{i \in I} f_i(A_1,...,A_m), \bigcup_{i \in I} f_i(B_1,...,B_m)\right) = \sup_{z \in \bigcup_{i \in I} f_i(A_1,...,A_m)} d\left(z, \bigcup_{i \in I} f_i(B_1,...,B_m)\right) \leq \eta - \lambda$. Interchanging the roles of $\bigcup_{i \in I} f_i(A_1,...,A_m)$ and $\bigcup_{i \in I} f_i(B_1,...,B_m)$ we also obtain that $d\left(\bigcup_{i \in I} f_i(B_1,...,B_m), \bigcup_{i \in I} f_i(A_1,...,A_m)\right) \leq \eta - \lambda$ and hence $h\left(\bigcup_{i \in I} f_i(A_1, A_2,...,A_m), \bigcup_{i \in I} f_i(B_1, B_2,...,B_m)\right) \leq \eta - \lambda < \eta$. Thus we obtain that $h(F_{\mathcal{S}^m}(A_1,...,A_m), F_{\mathcal{S}^m}(B_1,...,B_m)) = h\left(\overline{\bigcup_{i \in I} f_i(A_1, A_2,...,A_m)}, \overline{\bigcup_{i \in I} f_i(B_1, B_2,...,B_m)}\right) = h\left(\bigcup_{i \in I} f_i(A_1, A_2,...,A_m), \bigcup_{i \in I} f_i(B_1, B_2,...,B_m)\right) < \eta$.

c). If f_k is contractive or non-expansive then $Lip(f_k) \leq 1$ for every $k \in I$. From point a), $Lip(F_{\mathcal{S}}) \leq \sup_{k \in I} Lip(f_k) \leq 1$ and thus $F_{\mathcal{S}}$ is non-expansive.

Example 31. Let $X = [0,1]$ and the functions $f_n : [0,1] \times [0,1] \longrightarrow [0,1]$ defined by $f_n(x,y) = \frac{x+y}{2^{n+2}} + \frac{1}{2^{n+1}}$ for every $n \in \mathbb{N}$ which constitute a bounded family. Then for every $\eta > 0$ there exist $\delta = \frac{\eta}{3} > 0$ and $\lambda = \frac{\eta}{9} > 0$ such that $x_1, y_1, x_2, y_2 \in [0,1]$ with $|x_1 - x_2| < \eta + \delta$ and $|y_1 - y_2| < \eta + \delta$ implies $|f_n(x_1, y_1) - f_n(x_2, y_2)| \leq \frac{|x_1 - x_2| + |y_1 - y_2|}{2^{n+2}} < \frac{2\eta + 2\delta}{2^{n+2}} = \frac{2\eta + \frac{2\eta}{3}}{2^{n+2}} = \frac{8\eta}{3 \cdot 2^{n+2}} \leq \frac{8\eta}{9} = \eta - \frac{\eta}{9} = \eta - \lambda$ for every $n \in \mathbb{N}$. Hence the family of functions $(f_n)_{n \geq 0}$ is uniformly strong Meir-Keller. We consider now the $(GIIFS)$, $\mathcal{S}^2 = (X^2, (f_n)_{n \geq 0})$ and we obtain that the fractal operator $F_{\mathcal{S}^2} : \mathcal{B}(X)^2 \longrightarrow \mathcal{B}(X)$, $F_{\mathcal{S}^2}(B_1, B_2) = \overline{\bigcup_{k \geq 0} f_k(B_1, B_2)}$ is strong Meir-Keller and has a unique fixed point. We remark that the attractor of $\mathcal{S}^2$ is the interval $[0,1]$.

We can give now the existence and the uniqueness of the attractor of a generalized infinite iterated function system ($GIIFS$) formed by contractions or by a family of uniformly strong Meir-Keller functions.

Theorem 32. *Let* (X,d) *be a complete metric space,* $m \in \mathbb{N}^*$, *and* $\mathcal{S}^m = (X^m, (f_i)_{i\in I})$ *a* $(GIIFS)$, *where the family of functions* $(f_i)_{i\in I}$ *is bounded. Then the fractal operator* $F_{\mathcal{S}^m} : \mathcal{B}(X)^m \longrightarrow \mathcal{B}(X)$ *defined by* $F_{\mathcal{S}^m}(Y_1,...,Y_m) = \overline{\bigcup_{i\in I} f_i(Y_1,...,Y_m)}$ *has a unique fixed point, which is called the attractor of* $\mathcal{S}^m$, *in the following situations:*

a). f_i *is a contraction for every* $i \in I$ *with* $c = \sup_{i\in I} Lip(f_i) < 1$.

b). The family $(f_i)_{i\in I}$ *is uniformly strong Meir-Keeler.*

Proof: We can apply Theorems 6, 19 and 30 to the operator $F_{\mathcal{S}^m}$ and obtain in both cases that there exists a unique $A \in \mathcal{B}(X)$ such that $F_{\mathcal{S}^m}(A,A,...,A) = A$.

In the compact case, we can also obtain the existence and uniqueness of the attractor in the finite case for contractive functions.

Theorem 33. *Let* (X,d) *be a compact metric space,* $m \in \mathbb{N}^*$, *and* $\mathcal{S}^m = (X^m, (f_i)_{i\in I})$ *a* $(GIIFS)$, *where the family of functions* $(f_i)_{i\in I}$ *is bounded. If* f_i *is contractive for every* $i \in I$ *and* I *is finite, then the fractal operator* $F_{\mathcal{S}^m} : \mathcal{K}(X)^m \longrightarrow \mathcal{K}(X)$ *defined by* $F_{\mathcal{S}^m}(Y_1,...,Y_m) = \bigcup_{i\in I\ finite} f_i(Y_1,...,Y_m)$ *has a unique fixed point, which is called the attractor of* $\mathcal{S}^m$.

Proof: We can apply Theorems 19, 22 and 30 to obtain the existence and uniqueness of the attractor.

We remark that Theorem 33 from above is not valid if I is infinite as one can see from the example below.

Example 34. Let $X = [0,1]$, $m = 1$, $I = [0,1)$ and $f_i : [0,1] \longrightarrow [0,1]$ defined by $f_i(x) = ix$ for every $x \in [0,1]$ and $i \in [0,1)$. Then f_i is contractive since $|f_i(x) - f_i(y)| = |ix - iy| = i \cdot |x-y| < |x-y|$ for every $i \in [0,1)$. Let $Y_1 = \{0\}$. Then $f_i(0) = 0$ for every $i \in I$ which implies $F_{\mathcal{S}^1}(Y_1) = F_{\mathcal{S}^1}(\{0\}) = \overline{\bigcup_{i\in I} f_i(\{0\})} = \overline{\bigcup_{i\in I} \{0\}} = \{0\} = Y_1$. Let $Y_2 = [0,1]$. Then $f_i(Y_2) = [0,i)$ for every $i \in I$ which implies $F_{\mathcal{S}^1}(Y_2) = F_{\mathcal{S}^1}([0,1]) = \overline{\bigcup_{i\in I} f_i([0,1])} = \overline{\bigcup_{i\in I} [0,i)} = [0,1] = Y_2$. Hence $F_{\mathcal{S}^1}$ does not have a unique fixed point.

Open problem 2. In the conditions of Theorem 30, point b), does it result that $F_{\mathcal{S}^m}$ is Meir-Keeler?

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