Moduli Spaces of Rank–3 Vector Bundles on Hirzebruch Surfaces with $c_1 = \sigma + 2f$[1]

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Abstract

We study the nonemptyness, the irreducibility, the smothness and the unirationality of the moduli spaces of rank 3 stable vector bundles on a Hirzebruch surface with $c_1 = \sigma + 2f$.
Keywords: *vector bundles, moduli spaces, ruled surfaces.*
AMS Classification: 14F05, 14J60, 14J26

1. ***Introduction***

This paper is a natural sequel to [2] and [5].
In [2] is described the moduli spaces of rank-3 vector bundles with the first Chern class $c_1 = 0$ and $c_1 = f + \sigma$. In [5] is presented the moduli spaces of rank-3 vector bundles with the first Chern class $c_1 = \sigma$. In this paper we discuss the case when $c_1 = \sigma + 2f$ using the same techniques as in [2].

We denote by X the rational ruled surface with the ruling $\pi : X \longrightarrow \mathbb{P}^1$. We call f a fiber of π, and σ a section to π such that $\sigma^2 = -e$. We remind that the canonical divisor of X is $K_X = -2\sigma - (2+e)f$.
If $L = x\sigma + yf$ is an ample divisor, we note $r_L = \frac{y}{x}$.

As in [2], c_1 will be a divisor on X, c_2 an integer such that:

$$3c_2 - c_1^2 > 0,$$

and let $\mathcal{C}_1$, $\mathcal{C}_2$ be two adjacent chambers of type $(3, c_1, c_2)$.

We take V a rank-3 vector bundle L_1 stable (with $L_1 \in \mathcal{C}_1$), but L_2 unstable ($L_2 \in \mathcal{C}_2$) and the Harder-Narasimhan filtration of V with respect of L_2 given by:

$$0 \subset V_1 \subset ... \subset V_k \subset V.$$

By [2], we have that one of the next cases must happen:

1. $k = 1, F = c_1(V_1)$ and $i = rank(V_1) = 1$;

[1]Citation: Sterian A., *Moduli Spaces of Rank–3 Vector Bundles on Hirzebruch Surfaces with $c_1 = \sigma + 2f$*, "An. Univ. Spiru Haret, Ser. Mat.-Inform.", 12(1), pp. 13-23, 2016.

2. $k = 1, F = c_1(V_1)$ and $i = rank(V_1) = 2$;

3. $k = 2, F = c_1(V_2)$ and $i = rank(V_2) = 2$;

4. $k = 2, F = c_1(V_1)$ and $i = rank(V_1) = 1$.

From [2] we know that for every rank-3 vector bundle L_1-stable and L_2-unstable, there exist an integer $0 < i < 3$ and a divisor F such that:

1. $(3F - ic_1) \cdot L_1 < 0 < (3F - ic_1) \cdot L_2$;

2. $-4 \cdot (3c_2 - c_1^2) \leq (3F - ic_1)^2 < 0$

By [2] we have the next result:

Theorem 1.1(see [2]) *Let $\xi = (3F - c_1)$ a nonempty wall of type $(3; c_1, c_2)$ and let $\mathcal{C}$ be a chamber of type $(3; c_1, c_2)$ such that its closure $\bar{\mathcal{C}}$ intersects with W^ξ and that $L_1 \cdot \xi < 0$ for $L_1 \in \mathcal{C}$. Let L_0 be an ample divisor contained in $(\bar{\mathcal{C}} \cap W^\xi)$. Assume that V is a rank-3 bundle given by the nontrivial extension*

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow V \longrightarrow V' \longrightarrow 0,$$

where V' is L_0-semistable such that no L_0-destabilizing rank-1 subsheaf can be lifted to a subsheaf of V and V has the Chern class $c_1(V) = c_1$ and $c_2(V) = c_2$. If $c_1 \neq 0$ [mod 3] and if V' is L_1-semistable, then V is L_1-stable

In the remaining part of this paper we keep the structure adopted in [5].

2. ***The first case***

Let $V_1 = \mathcal{O}_X(F)$ and let $W = V/V_1$. By Proposition 1.2-[2], W is torsion free and L_2 semistable. We shall estimate the number of moduli of those V's coming from the next exact sequences:

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow V \longrightarrow W \longrightarrow 0 \qquad (1)$$

and

$$0 \longrightarrow W \longrightarrow W^{**} \longrightarrow Q \longrightarrow 0, \qquad (2)$$

where Q is a sheaf supported on some 0-cycles in X. We have the next result from [2]:

Proposition 2.1(see [2]) *Fix a divisor F. Let $\xi = 3F - c_1$, and let*

$$d_\xi(c_1, c_2) = -\frac{3c_2 - c_1^2}{3} + 2 + \frac{\xi^2}{6} + \frac{K_X \cdot \xi}{2}.$$

Then, #(moduli of V)$\leq (6c_2 - 2c_1^2 - 8) + d_\xi(c_1, c_2)$.

Lemma 2.2 *Let* $\xi = (3F - c_1)$.
If $c_1 = \sigma + 2f$ *then* $d_\xi(c_1, c_2) < 0$, *unless possibly either of the following cases:*

$$a) \xi = 2\sigma - (3c_2 - 4)f$$

$$b) e = 0 : \xi = \frac{-3c_2 + 7}{2}\sigma + f \ or \ \xi = \frac{-3c_2 + 4}{2}\sigma + f.$$

Proof. Let $\xi = x\sigma - yf$.
The inequality $\xi L_1 < 0 < \xi L_2$ implies that x and y have the same sign and none of them are zero. We are now able to evaluate the expression of

$$d_\xi(c_1, c_2) = -\frac{3c_2 - c_1^2}{3} + 2 + \frac{-x(ex - 3e + 6) - 2y(x - 3)}{6}.$$

If $c_2 = \sigma + 2f$, then $F = \frac{x+1}{3}\sigma - \frac{y-2}{3}f$, $x \equiv 2[3]$ and $y \equiv 2[3]$.
Let suppose that $x > 0$. This implies that $x \geq 2$ and $y \geq 2$. For $x \geq 5$, it's obvious that $d_\xi(c_1, c_2) < 0$.
If $x = 2$, then

$$d_\xi(c_1, c_2) = \frac{(e + y) - (3c_2 - c_1^2)}{3}.$$

Since $-4(3c_2 - c_1^2) \leq (3F - ic_1)^2 = \xi^2 = (x\sigma - yf)^2 = -4(e + y)$, we have $d_\xi(c_1, c_2) \leq 0$ with equality if and only if $x = 2$ and $y = 3c_2 - 4$, and so

$$\xi = 2\sigma - (3c_2 - 4)f.$$

If $x < 0$, then $x \leq -1$ and $y \leq -1$.
If $e \geq 1$, then $d_\xi(c_1, c_2) < 0$.
If $e = 0$, then

$$d_\xi(c_1, c_2) = -\frac{3c_2 - c_1^2}{3} + 2 + \frac{-6x - 2y(x - 3)}{6}.$$

Since $y \leq -4$,

$$\frac{-6x - 2y(x - 3)}{6} \leq \frac{-6x + 8(x - 3)}{6} < -4,$$

and so $d_\xi(c_1, c_2) \leq -2$.
If $y = -1$, then

$$d_\xi(c_1, c_2) = \frac{-3c_2 + 4}{3} - \frac{2x}{3} + 1 = \frac{-3c_2 + 7 - 2x}{3} \leq 1,$$

since $-2x = y \leq 3c_2 - 4$.
We observe here that for $x < 0$, $-2x \equiv 2[3]$, thus $-2x$ will take the same values as $y > 0$. Since from the first part of the proof we saw that for $y > 0$,

$y \leq 3c_2 - 4$, it results $-2x \leq 3c_2 - 4$.
It follows that

$$d_\xi(c_1, c_2) = 0 \text{ if } \xi = -\frac{3c_2 - 7}{2}\sigma + f$$

and

$$d_\xi(c_1, c_2) = 1 \text{ if } \xi = -\frac{3c_2 - 4}{2}\sigma + f.$$

Remark 2.3
For $c_1 = \sigma + 2f$ and $\xi = 2\sigma - (3c_2 - 4)f$, a small computation like in Remark 3.3 from [5] shows that V sits in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(\sigma - (c_2 - 2)f) \longrightarrow V \longrightarrow \mathcal{O}(\frac{c_2 f}{2}) \oplus \mathcal{O}_X(\frac{c_2 f}{2}) \longrightarrow 0$$

3. *The second case*

In this section we estimate the number of moduli of those rank-3 vector bundles V's with $c_1 = \sigma + 2f$ which satisfy case 2). We will follow the approach of section 2.3 from [2]:
Let be V a rank-3 vector bundle given by the exact sequence:

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z, \tag{3}$$

where V_1 is a L_2 semistable rank-2 bundle. Dualizing the above sequence, we have:

$$0 \longrightarrow \mathcal{O}_X(F - c_1) \longrightarrow V^* \longrightarrow W \longrightarrow 0 \tag{4}$$

where W is torsion free and $W^* = V_1$. Since V^* is L_1 stable but L_2 unstable, the Harder-Narasimhan filtration of V^* with respect of L_2 is

$$0 \subset \mathcal{O}_X(F - c_1) \subset V^*.$$

In this case, $c_1(V^*) = -c_1$ and $c_2(V^*) = c_2$. V^* is now in the same hypothesis as the above section, so applying Proposition 3.1 we get:

$$\xi = 3(F - c_1) - (-c_1) = 3F - 2c_1$$

and

$$\#(\text{moduli of } V) = \#(\text{moduli of } V^*) \leq (6c_2 - c_1^2 - 8) + d_\xi(-c_1, c_2).$$

Lemma 3.1 *Let* $\xi = 3F - 2c_1$.
If $c_1 = \sigma + 2f$, *then* $d_\xi(-c_1, c_2) \leq 0$. *Moreover* $d_\xi(-c_1, c_2) = 0$ *if and only if*
a)$\xi = \sigma - \frac{3c_2 - 7}{2} f$ *or*
b)$e = 0$ *and* $\xi = -(3c_2 - 4)\sigma + 2f$.

Proof. Let $\xi = 3F - 2c_1 = x\sigma - yf$. From the same argument as in the above lemma x and y have the same sign and can not be zero. By Proposition 2.1,

$$d_\xi(-c_1, c_2) = -c_2 + \frac{6-3x-e}{3} + \frac{(xe+2y)(3-x)}{6}.$$

Let $c_1 = \sigma + 2f$, then

$$F = \frac{x+2}{3}\sigma - \frac{y-4}{3},\ x \equiv 1[3] \text{ and } y \equiv 1[3].$$

If $x > 3$, then $x \geq 4$, $y \geq 1$ and $d_\xi(-c_1, c_2) \leq 0$.
Assume that $x < 3$, then $x = 1$ or $x \leq -2$. If $x = 1$, then

$$d_\xi(-c_1, c_2) = -c_2 + \frac{2y+7}{3}.$$

Since $c_1(V_1) = F = \sigma - \frac{y-4}{3}f$ and $c_2(V_1) \leq c_2 - F(c_1 - F)$, we get

$$c_2(V_1) \leq c_2 - \frac{y+2}{3}.$$

Since V_1 is L_2 stable, from lemma 1.10 of [4] we have:

$$r_{L_2} \leq 2c_2 - \frac{y+8}{3} + e,$$

and from the fact that $\xi \cdot L_2 > 0$, we obtain $r_{L_2} > e + y$.
Therefore

$$e + y < r_{L_2} \leq c_2 - \frac{y+2}{3},$$

and so

$$\frac{2y}{3} < c_2 - \frac{4}{3}.$$

Since $d_\xi(-c_1, c_2) = -c_2 + \frac{2y}{3} + \frac{7}{3} < -c_2 + (c_2 - \frac{4}{3} + \frac{7}{3} = 1)$, it follows

$$d_\xi(-c_1, c_2) \leq 0,$$

with equality if and only if

$$y = \frac{3c_2 - 7}{2}.$$

If $x \leq -2$, then $y \leq -2$. For $e \geq 1$ we get $d_\xi(-c_1, c_2) < 0$. Let $e = 0$. If $y \leq -5$, then $d_\xi(-c_1, c_2) < 0$.
If $y = -2$, then

$$d_\xi(-c_1, c_2) = -\frac{3c_2 - c_1^2}{3} - \frac{x}{3} \leq 0,$$

since $-4(3c_2 - c_1^2) \leq \xi^2 = 4x$, and with equality for $x = -(3c_2 - 4)$.

Remark 3.2
If $c_1 = \sigma + 2f$ and $\xi = \sigma - \frac{3c_2-7}{2}f$, then $F = \sigma - \frac{c_2-5}{2}f$. Since

$$e + \frac{3c_2 - 7}{2} = e + y \leq r_{L_2} < e + 2c_2(V_1) - \frac{5 - c_2}{2},$$

it follows $\frac{c_2-1}{2} < c_2(V_1)$, and thus $\frac{c_2+1}{2} \leq c_2(V_1)$. From the first exact sequence we get $c_2(V_1) + l(Z) = c_2 - F(c_1 - F)$, and so

$$c_2(V_1) + l(Z) = \frac{c_2 + 1}{2}.$$

Therefore, $c_2(V_1) = \frac{c_2+1}{2}$, $l(Z) = 0$ and V sits in the exact sequence

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2 - 1}{2}) \longrightarrow 0. \tag{5}$$

As in [5], one can demonstrate that V_1 sits in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(\sigma - (c_2 - 2)f) \longrightarrow V_1 \longrightarrow \mathcal{O}_X(\frac{1 + c_2}{2}f) \longrightarrow 0. \tag{6}$$

Since:

$$V_1/\mathcal{O}_X(\sigma - (c_2 - 2)f) = \mathcal{O}(\frac{1 + c_2}{2}f) \text{ and } V/V_1 = \mathcal{O}_X(\frac{c_2 - 1}{2}f),$$

it follows that $V/\mathcal{O}_X(\sigma - (c_2 - 2)f)$ is given by the extension

$$0 \longrightarrow \mathcal{O}(\frac{1 + c_2}{2}f) \longrightarrow V/\mathcal{O}_X(\sigma - (c_2 - 2)f) \longrightarrow \mathcal{O}_X(\frac{c_2 - 1}{2})f \longrightarrow 0.$$

This sequence splits because $Ext^1(\mathcal{O}_X(\frac{c_2-1}{2}f), \mathcal{O}_X(\frac{c_2+1}{2}f))) = 0$, so finally we obtain $V/\mathcal{O}_X(\sigma - (c_2 - 2)f) = \mathcal{O}(\frac{c_2-1}{2}f) \oplus \mathcal{O}(\frac{c_2+1}{2}f)$, and V comes from the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(\sigma - (c_2 - 2)f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2 - 1}{2}f) \oplus \mathcal{O}_X(\frac{c_2 + 1}{2}f) \longrightarrow 0.$$

Proposition 3.3 *Let c_2 be odd.*
If $c_1 = \sigma + 2f$, then for generic bundles V_1 in (6), generic extensions V in (5) are L' stable, where L' is any ample divisor contained in the chamber whose upper wall is $W^{(\sigma - \frac{3c_2-7}{2}f)}$. Moreover, all such V's are parametrized by a smooth rational variety of dimension $(6c_2 - 2c_1^2 - 8)$.

Proof. The proof is the same as the proof of Proposition 4.3 in [5].

4. **The third case**

We recall the set up of section 2.4. in [2]. Let $c_1(V_1) = F_1$ and $c_1(V_2) = F_2 = F$. Therefore, we have the exact sequences:

$$0 \longrightarrow V_2 \longrightarrow V \longrightarrow \mathcal{O}_X(c_1 - F_2) \otimes I_{Z_2} \longrightarrow 0 \tag{7}$$

$$0 \longrightarrow \mathcal{O}_X(F_1) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(F_2 - F_1) \otimes I_{Z_1} \longrightarrow 0. \tag{8}$$

From the proof of Lemma 1.4 in [2], we have

$$(2F_1 - F_2) \cdot L_1 < 0 < (2F_1 - F_2) \cdot L_2,$$

and so from the second exact sequence

$$c_1(V_2)^2 - 4c_2(V_2) \leq (2F_1 - F_2)^2 < 0.$$

To start with, we remind the Proposition 2.20 from [2].

Proposition 4.1(see [2]) *Fix $F = F_2 = c_1(V_2)$. Let $\eta = (3F - 2c_1)$ and let*

$$d^*_\eta(c_1, c_2) = -\frac{2(c_2 - c_1^2)}{3} + 3 + \frac{(2F_1 - F_2)^2}{4} + \frac{(2F_1 - F_2) \cdot K_X}{2} + \frac{\eta^2}{12} + \frac{\eta \cdot K_X}{2}.$$

Then,

$$\#(\textit{moduli of } V) \leq (6c_2 - 2c_1^2 - 8) + d^*_\eta(c_1, c_2).$$

Lemma 4.2 *Put $\eta = (3F - 2c_1)$*
If $c_1 = \sigma + 2f$, then $d^*_\eta(c_1, c_2) < 0$, unless possibly either of the following:

$$\eta = \sigma - \frac{3c_2 - 7}{2} f \quad \textit{or} \quad \eta = \sigma - \frac{3c_2 - 4}{2} f.$$

Proof. Put $(2F_1 - F_2) = x_1\sigma - y_1 f$ and $\eta = x_2\sigma - y_2 f$. From the same argument as in the proof of Lemma 2.21 in [2], x_i and y_i must have the same sign and neither of them cannot be zero. By the above proposition,

$$d^*_\eta(c_1, c_2) = -\frac{2(c_2 - c_1^2)}{3} + 3 + d_1 + d_2,$$

where

$$d_1 = \frac{(2F_1 - F_2)^2}{4} + \frac{(2F_1 - F_2) \cdot K_X}{2} = \frac{-x_1(ex_1 - 2e + 4) - 2y_1(x_1 - 2)}{4},$$

$$d_2 = \frac{\eta^2}{12} + \frac{\eta \cdot K_X}{2} = \frac{-x_2(ex_2 - 6e + 12) - 2y_2(x_2 - 6)}{12}.$$

Also, from the proof of the Lemma 2.21 in [2] we know that

$$d_1 - \frac{3c_2 - c_1^2}{3} \le \frac{\eta^2}{12} - 1.$$

If $c_1 = \sigma + 2f$, then $F = \frac{x_2+2}{3}\sigma - \frac{y_1-4}{3}f$, $x_2 \equiv 1[3]$ and $y_2 \equiv 1[3]$.
If $x_2 \ge 7$, then it's obvious that $d_\eta^*(c_1, c_2) < 0$.
If $x_2 = 4$, then $d_2 = \frac{2e+y_2}{3} - 4$, and since $-4(3c_2 - c_1^2) \le \eta^2 = -8(2e + y_2)$, we obtain

$$d_2 \le \frac{3c_2 - c_1^2}{6} - 4,$$

and so $d_\eta^*(c_1, c_2) < 0$.
If $x_2 = 1$, then $d_2 = \frac{5(e+2y_2}{12} - 1$. As in case of the lemma, using the proof of Theorem 3.1 in [1] we obtain $r_{L_0} < e + \frac{3c_2-1}{2}$. Since $r_{L_0} = e + y_2$, we get $y_2 < \frac{3c_2-1}{2}$. Since $y_2 \equiv 1[3]$, $2y_2 \equiv 2[3]$ and $2y_2 \le 3c_2 - 4$. As in the above case, for $x_1 \ne 1$, $d_1 \le -2$ and $d_\eta^*(c_1, c_2) < 0$.
If $x_1 = 1$, $d_1 - \frac{(3c_2-c_1^2)}{3} \le \frac{\eta^2}{12} - 1$, and it follows

$$d_\eta^*(c_1, c_2) = -\frac{(c_2 - c_1^2)}{3} + 3 + (\frac{\eta^2}{12} - 1) + (\frac{(5e + 10y_2)}{12} - 1) \le$$

$$\le \frac{2y_1 + 7 - 3c_2}{3} \le \frac{3c_2 - 4 + 7 - 3c_2}{3} = 1.$$

We conclude that $d_\eta^*(c_1, c_2) < 0$ unless when

$$y_2 = \frac{3c_2 - 7}{2} \text{ or } y_2 = \frac{3c_2 - 4}{2}.$$

If $x_2 < 0$, then $x_2 \le -2$ and $y_2 \le -2$. If $e \ge 1$, then $d_\eta^*(c_1, c_2) < 0$. If $e = 0$, then $d_\eta^*(c_1, c_2) < 0$, unless possibly $y_2 = -2$ and $x_2 = -3c_2 + 4$. In this case, $d_\eta^*(c_1, c_2) = 0$.

Remark 4.3
If $c_1 = \sigma + 2f$ and $\eta = \sigma - \frac{3c_2-4}{2}f$, then we get $c_1(V_2) = F = \sigma - \frac{c_2-4}{2}f$. By the exact sequence (7), we have $c_2(V_2) + l(Z_2) = c_2 - F(c_1 - F) = \frac{c_2}{2}$. On the other side, since V is L_0-semistable, for $L_0 \in W^\eta$, $\frac{c_1(V_2)\cdot L_0}{2} = \frac{c_1 \cdot L_0}{3}$, V_2 must be L_0-semistable, and thus by the Lemma 1.10 in [4], we get:

$$e + \frac{3c_2 - 4}{2} = r_{L_0} \le 2c_2(V_2) + e + \frac{c_2 - 4}{2},$$

and so $\frac{c_2}{2} \le c_2(V_2)$. We obtain $c_2(V_2) = \frac{c_2}{2}$ and $l(Z_2) = 0$. As in the above remark, we show that V_2 sits in the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(\sigma - (c_2 - 2)f) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(\frac{c_2}{2})f \longrightarrow 0,$$

and that V is given by the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(\sigma - (c_2-2)f) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(\frac{c_2}{2}f) \oplus \mathcal{O}_X(\frac{c_2}{2}f) \longrightarrow 0.$$

We conclude that the bundles V are exactly those in subsection 3).

Remark 4.4
Let $c_1 = \sigma + 2f$ and $\eta = \sigma - \frac{3c_2-7}{2}$, then $c_1(V_2) = F_2 = F = \sigma - \frac{c_2-5}{2}f$. We can verify that $F_1 = \sigma - (c_2-2)f$, $c_2(V_2) = \frac{c_2+1}{2}$, and that Z_1 and Z_2 are empty.
Therefore, our exact sequence will be:

$$0 \longrightarrow V_2 \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2-1}{2}f) \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{O}(\sigma - (c_2-2)f) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(\frac{c_2+1}{2}f) \longrightarrow 0.$$

5. ***The fourth case***

We begin this section, reminding the hypothesis from [2]-2.5. Let $F_1 = c_1(V_1) = F$ and $F_2 = c_1(V_2)$. The rank-3 vector bundle satisfying case d are sitting in the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(F_1) \longrightarrow V \longrightarrow V/\mathcal{O}_X(F_1) \longrightarrow 0,$$

where $V/\mathcal{O}_X(F_1)$ is coming from the exact sequence:

$$0 \longrightarrow V_2/\mathcal{O}_X(F_1) \longrightarrow V/\mathcal{O}_X(F_1) \longrightarrow \mathcal{O}_X(c_1 - F_2) \otimes I_Z \longrightarrow 0,$$

with the remark from the proof of the Lemma 1.4 that:

$$(2F_2 - F_1 - c_1) \cdot L_1 < 0 < (2F_2 - F_1 - c_1) \cdot L_2.$$

Dualizing the second exact sequence it follows:

$$0 \longrightarrow [V/\mathcal{O}_X(F_1)]^* \longrightarrow V^* \longrightarrow \mathcal{O}_X(-F_1) \otimes I_{Z_2} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(F_2 - c_1) \longrightarrow [V/\mathcal{O}_X(F_1)]^* \longrightarrow \mathcal{O}_X(F_1 - F_2) \otimes I_{Z_1} \longrightarrow 0.$$

Remark that V^* is L_1-stable but L_2 unstable and

$$2(F_2 - c_1) - c_1([V/\mathcal{O}_X(F_1)]^*) = (2F_2 - F_1 - c_1).$$

From the subsection $d.$ of [2] we know that:

$$\#(\text{moduli of } V) = \#(\text{moduli of } \ V^*) \leq (6c_2 - 2c_1^2 - 8) + d_\eta^*(-c_1, c_2),$$

where

$$\eta = 3c_1([V/\mathcal{O}_X(F_1)]^*) - 2c_1(V^*) = (3F_1 - c_1)$$

and

$$\begin{aligned} d^*_\eta(-c_1, c_2) &= -\tfrac{2(3c_2-c_1^2)}{3} + 3 + \tfrac{(2F_2-F_1-c_1)^2}{4} + \tfrac{(2F_2-F_1-c_1)\cdot K_X}{2} + \\ &+ \tfrac{\eta^2}{12} + \tfrac{\eta\cdot K_X}{2}. \end{aligned}$$

Lemma 5.1 *Put* $\eta = 3F_1 - c_1$.
If $c_1 = \sigma + 2f$, *then* $d^*_\eta(-c_1, c_2) < 0$ *unless possibly either of the following*
a)$\eta = 2\sigma - (3c_2 - 4)f$;
b)$e = 0, \eta = -\frac{3c_2-4}{2}\sigma + f$ *or* $\eta = -\frac{3c_2-7}{2} + f$.

Proof.
Let $c_1 = \sigma + 2f$, then

$$F_1 = \frac{x_2+1}{3}\sigma - \frac{y_2-2}{3}f,$$

and so

$$x_2 \equiv 2[3] \text{ and } y_2 \equiv 2[3].$$

As in the proof of Lemma 6.1 in [5], if $x_2 \leq 5$, $d^*_\eta(-c_1, c_2) \leq -3$. and for $x_2 = 1$,$d^*_\eta(-c_1, c_2) \leq 0$, with equality if and only if $y_2 = 3c_2 - 4$. If $x_2 < 0$, then $x_2 \leq -1$ and $y_2 \leq -1$. For $e \geq 1$, we have always $d^*_\eta(-c_1, c_2) \leq 0$. If $e = 0$ then $d^*_\eta(-c_1, c_2) \leq 1$, and we have:

$$a) d^*_\eta(-c_1, c_2) = 1 \text{ if and only if } y_2 = -1 \text{ and } x_2 = -\frac{3c_2-4}{2};$$

$$b) d^*_\eta(-c_1, c_2) = 0 \text{ if and only if } y_2 = -1 \text{ and } x_2 = -\frac{3c_2-7}{2}.$$

Remark 5.2
If $c_1 = \sigma + 2f$ and $\eta = 2\sigma - (3c_2 - 4)f$, then $F_1 = \sigma - (c_2 - 2)f$, and like in the first part of the remark we show that V sits in the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(\sigma - (c_2-2)f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2}{2}f) \oplus \mathcal{O}_X(\frac{c_2}{2}f) \longrightarrow 0.$$

6. *Moduli spaces of rank–3 vector bundles with* $c_1 = \sigma + 2f$

We are now able to give the characterization of the moduli spaces of rank–3 vector bundles with $c_1 = \sigma + 2f$:

Theorem 6.1 *Let* c_2 *an even integer. If* $c_1 = \sigma + 2f$ *then:*

1. *For* $c_2 < 2$, *all the moduli spaces* $\mathcal{M}_L(3, \sigma + 2f, c_2)$ *are empty.*

2. *For* $c_2 \geq 2$ *and* $r_L \geq e+\frac{3c_2-4}{2}$ *or* $r_L \leq \frac{2}{3c_2-4}$, *the moduli space* $\mathcal{M}_L(3,\sigma+2f,c_2)$ *is empty.*

3. *For* $c_2 \geq 2$ *and* $\frac{2}{3c_2-4} < r_L < e+\frac{3c_2-4}{2}$, *then the moduli space* $\mathcal{M}_L(3,\sigma+2f,c_2)$ *is irreducible, smooth and unirrational with dimension* $(6c_2+2e-16)$. *Moreover, in this case a generic bundle in* $\mathcal{M}_L(3,\sigma+2f,c_2)$ *sits in an exact sequence:*

$$0 \longrightarrow \mathcal{O}_X(\sigma-(c_2-2)f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2 f}{2}) \oplus \mathcal{O}_X(\frac{c_2 f}{2}) \longrightarrow 0. \quad (9)$$

Proof. The idea of the proof is the same as in the case of Theorem 3.2 in [2].

Theorem 6.2 *Let* c_2 *an odd integer.*
If $c_1 = \sigma+2f$ *then:*

1. *For* $c_2 < 3$, *then all the moduli spaces* $\mathcal{M}_L(3,\sigma+2f,c_2)$ *are empty.*

2. *For* $c_2 \geq 3$ *and* $r_L \geq e+\frac{3c_2-7}{2}$ *or* $r_L \leq \frac{2}{3c_2-7}$, *the moduli space* $\mathcal{M}_L(3,\sigma+2f,c_2)$ *is empty.*

3. *For* $c_2 \geq 3$ *and* $\frac{2}{3c_2-7} < r_L < e+\frac{3c_2-7}{2}$, *then the moduli space* $\mathcal{M}_L(3,\sigma+2f,c_2)$ *is irreducible, smooth and rational with dimension* $(6c_2+2e-16)$. *Moreover, in this case a generic bundle in* $\mathcal{M}_L(3,\sigma+2f,c_2)$ *sits in an exact sequence:*

$$0 \longrightarrow \mathcal{O}_X(\sigma-(c_2-2)f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2-1}{2}f) \oplus \mathcal{O}_X(\frac{c_2+1}{2}f) \longrightarrow 0.$$

Proof. The proof is the same as the proof of the Theorem 3.4 in [2]

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