Remarks on Robertson–Walker Warped Products Equations[1]

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Abstract

The aim of this paper is to investigate, primarily small dimension, multiply generalized warped products $M = I \times_{f_1} F_1 \times_{f_2} \cdots \times_{f_m} F_m$ *which are Einstein spaces, where* $I = (t_1, t_2)$ *is an interval with* $-\infty \leq t_1 \leq t_2 \leq \infty$ *and* $\dim F_i = k_i \geq 1$ *for every* $i \in \{1, ..., m\}$*. If* $M = I \times_{f_1} F_1 \times_{f_2} ... \times_{f_m} F_m$ *we will say that* M *is of type* $(1, k_1, ..., k_m)$ *and* $\dim M = 1 + k_1 + ... k_m$*. Thus, considering the multiply generalized Einstein warped product equations, we compute, exactly or in the parametrized form, the warping functions in the following cases:*

a) M *is of type* $(1, 1)$.
b) M *is of type* $(1, 2)$ *or* $(1, 1, 1)$.
c) M *is* $(1, 3)$, $[(1, 1, 2)$ *and Ricci flat*$]$ *or* $(1, 1, 1, 1)$.
d) M *is* $(1, 4)$, $(1, 1, 3)$ *or* $(1, 1, 1, 1, 1)$.
e) M *is* $(1, \underbrace{1, 1, ..., 1}_{p\ times})$ *with* $p \geq 5$ *or* $[(1, 1, k)$ *and Ricci flat*$]$ *with* $k \geq 4$.
f) Equal warping function.

Keywords: *Einstein space, multiply warped product, warping function.*

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1. ***Introduction***

The notion of warped product was introduced in ([3]) and it served to give examples of new Riemannian manifolds. Warped products also appear in general relativity as models for space-time, especially the Robertson–Walker spaces or their generalizations, which in some cases can be the solutions of Einstein's field equations ([1], [2], [5], [6], [7], [8], [9]). According to ([4]) we have the following two definitions:

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Definition 1.1. Let (B, g_B) and $((F_i, g_{F_i}))_{i=\overline{1,m}}$ be pseudo-Riemannian manifolds and $f_i : B \longrightarrow (0, \infty)$ smooth functions for every $i \in \{1, ..., m\}$, $m \geq 1$. The *multiply warped product* manifold is the product manifold $M = B \times F_1 \times ... \times F_m$ furnished with the metric tensor $g = g_B \oplus f_1^2 g_{F_1} \oplus ... \oplus f_m^2 g_{F_m}$.

Definition 1.2. If B is an open connected interval I of the form $I = (t_1, t_2)$ equipped with the negative definited metric $g_B = -dt^2$, where $-\infty \leq t_1 \leq t_2 \leq \infty$ and (F_i, g_{F_i}) is Riemannian for every $i \in \{1, ..., m\}$, then the multiply warped product (M, g) is called a *multiply generalized Robertson-Walker space-time.*

We want to investigate the case when a multiply generalized Robertson-Walker space-time is an Einstein space. In ([4]) are given the Einstein equations:

Theorem 1.1. *([4]) Let $M = I \times_{f_1} F_1 \times_{f_2} ... \times_{f_m} F_m$ be a multiply generalized Robertson-Walker space-time with the metric $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus ... \oplus f_m^2 g_{F_m}$. The space-time (M, g) is Einstein with Ricci curvature λ (i.e. $Ric_M = \lambda g$) if and only if the following conditions are satisfied for every $i \in \{1, ..., m\}$:*

$$\begin{cases} \text{Each fiber } (F_i, g_{F_i}) \text{ is Einstein with Ricci curvature } \lambda_{F_i} \\ \sum\limits_{i=1}^{m} k_i \frac{f_i''}{f_i} = \lambda \\ \lambda_{F_i} + f_i f_i'' + (k_i - 1)(f_i')^2 + f_i f_i' \sum\limits_{j=1, j \neq i}^{m} k_j \frac{f_j'}{f_j} = \lambda f_i^2. \end{cases} \tag{1}$$

where $k_i = \dim F_i$ and f_i', f_i'' represent the first and the second derivatives of f_i for every $i \in \{1, ..., m\}$.

Proposition 1.1. *([4]) Let $M = I \times_{f_1} F_1 \times_{f_2} ... \times_{f_m} F_m$ be a multiply generalized Robertson-Walker space-time with the metric $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus ... \oplus f_m^2 g_{F_m}$. If the space time (M, g) has constant scalar curvature τ_M, then each fiber (F_i, g_{F_i}) has constant scalar curvature τ_{F_i} for every $i \in \{1, ..., m\}$.*

Corollary 2.1. *([4]) Let $M = I \times_{f_1} F_1 \times_{f_2} ... \times_{f_m} F_m$ be a multiply generalized Robertson-Walker space-time with the metric $g = -dt^2 \oplus f_1^2 g_{F_1} \oplus ... \oplus f_m^2 g_{F_m}$ and, moreover, we suppose that (M, g) is Einstein. Then each fiber (F_i, g_{F_i}) has constant scalar curvature τ_{F_i} for every $i \in \{1, ..., m\}$.*

2. ***Main results***

The aim of this paper is to solve some particular forms of the system (1), the general solution of it being very difficult to give. We will primarily consider small dimension of the ambient space.

a) The case when $\dim M = 2$.

I) In this case $M = I \times_{f_1} F_1$ with $\dim F_1 = k_1 = 1$. Thus, the system (1) becomes:

$$\begin{cases} \frac{f_1''}{f_1} = \lambda \\ f_1 f_1'' = \lambda f_1^2 \end{cases} \quad (2)$$

We obtain that $f_1'' = \lambda f_1$. Hence:
i). If $\lambda > 0$, then $f_1(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$, $c_1, c_2 \in \mathbb{R}$.
ii). If $\lambda = 0$, then $f_1(x) = c_1 x + c_2$, $c_1, c_2 \in \mathbb{R}$.
iii). If $\lambda < 0$, then $f_1(x) = c_1 \cos(\sqrt{-\lambda} x) + c_2 \sin(\sqrt{-\lambda} x)$, $c_1, c_2 \in \mathbb{R}$.
b) The case when $\dim M = 3$.
II) $M = I \times_{f_1} F_1$ with $\dim F_1 = k_1 = 2$. Thus, the system (1) becomes:

$$\begin{cases} \frac{f_1''}{f_1} = \frac{\lambda}{2} \\ \lambda_{F_1} + f_1 f_1'' + \left(f_1'\right)^2 = \lambda f_1^2 \end{cases} \quad (3)$$

We obtain that $f_1'' = \frac{\lambda}{2} f_1$. Hence:

i) If $\lambda > 0$, then $f_1(x) = c_1 e^{\sqrt{\frac{\lambda}{2}} x} + c_2 e^{-\sqrt{\frac{\lambda}{2}} x}$, $c_1, c_2 \in \mathbb{R}$. Verifying also the second equation we obtain the supplementary condition: $c_1 c_2 = \frac{\lambda_{F_1}}{2\lambda}$.

ii) If $\lambda = 0$, then $f_1(x) = c_1 x + c_2$, $c_1, c_2 \in \mathbb{R}$. Verifying also the second equation we obtain the supplementary condition: $c_1^2 = -\lambda_{F_1}$. So, if $\lambda_{F_1} > 0$, we don't have any solution in this case.

iii) If $\lambda < 0$, then $f_1(x) = c_1 \cos\left(\sqrt{-\frac{\lambda}{2}} x\right) + c_2 \sin\left(\sqrt{-\frac{\lambda}{2}} x\right)$, $c_1, c_2 \in \mathbb{R}$. Verifying also the second equation we obtain the supplementary condition: $c_1^2 + c_2^2 = \frac{2\lambda_{F_1}}{\lambda}$. So, if $\lambda_{F_1} > 0$, we don't have any solution in this case.

III) $M = I \times_{f_1} F_1 \times_{f_2} F_2$ with $\dim F_1 = k_1 = 1$ and $\dim F_2 = k_2 = 1$. Thus, the system (1) becomes:

$$\begin{cases} \frac{f_1''}{f_1} + \frac{f_2''}{f_2} = \lambda \\ \frac{f_1''}{f_1} + \frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} = \lambda \\ \frac{f_2''}{f_2} + \frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} = \lambda \end{cases} \quad (4)$$

We obtain that $\frac{f_1''}{f_1} = \frac{f_2''}{f_2} = \frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} = \frac{\lambda}{2}$. Hence:

i) If $\lambda > 0$, then $f_1(x) = c_1 e^{\sqrt{\frac{\lambda}{2}} x} + c_2 e^{-\sqrt{\frac{\lambda}{2}} x}$ and $f_2(x) = d_1 e^{\sqrt{\frac{\lambda}{2}} x} + d_2 e^{-\sqrt{\frac{\lambda}{2}} x}$, $c_1, c_2, d_1, d_2 \in \mathbb{R}$. Verifying also the condition $\frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} = \frac{\lambda}{2}$, we obtain that $c_1 d_2 + c_2 d_1 = 0$.

ii) If $\lambda = 0$, then $f_1(x) = c_1 x + c_2$ and $f_2(x) = d_1 x + d_2$, $c_1, c_2, d_1, d_2 \in \mathbb{R}$. Verifying also the condition $\frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} = \frac{\lambda}{2}$, we obtain that $c_1 d_1 = 0$.

iii) If $\lambda < 0$, then $f_1(x) = c_1 \cos\left(\sqrt{-\frac{\lambda}{2}}x\right) + c_2 \sin\left(\sqrt{-\frac{\lambda}{2}}x\right)$ and $f_2(x) = d_1 \cos\left(\sqrt{-\frac{\lambda}{2}}x\right) + d_2 \sin\left(\sqrt{-\frac{\lambda}{2}}x\right)$, $c_1, c_2, d_1, d_2 \in \mathbb{R}$. Verifying also the condition $\frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2} = \frac{\lambda}{2}$, we obtain that $c_1 d_1 + c_2 d_2 = 0$.

c) The case when $\dim M = 4$.

IV) $M = I \times_{f_1} F_1$ with $\dim F_1 = k_1 = 3$. Thus, the system (1) becomes:

$$\begin{cases} 3\frac{f_1''}{f_1} = \lambda \\ \lambda_{F_1} + f_1 f_1'' + 2\left(f_1'\right)^2 = \lambda f_1^2 \end{cases} \quad (5)$$

Thus from equation (1) we obtain that $f_1'' = \frac{\lambda}{3} f_1$ and together with equation (2) we obtain $\lambda_1 + 2\left(f_1'\right)^2 = \frac{2\lambda}{3} f_1^2$.

i) $\lambda > 0 \Longrightarrow f_1(x) = c_1 e^{\sqrt{\frac{\lambda}{3}}x} + c_2 e^{-\sqrt{\frac{\lambda}{3}}x}$, $c_1, c_2 \in \mathbb{R}$. Then $f_1'(x) = \sqrt{\frac{\lambda}{3}}\left(c_1 e^{\sqrt{\frac{\lambda}{3}}x} - c_2 e^{-\sqrt{\frac{\lambda}{3}}x}\right)$. Hence $\lambda_{F_1} + \frac{2\lambda}{3}\left(c_1^2 e^{2\sqrt{\frac{\lambda}{3}}x} + c_2^2 e^{-2\sqrt{\frac{\lambda}{3}}x} - 2c_1c_2\right) = \frac{2\lambda}{3}\left(c_1^2 e^{2\sqrt{\frac{\lambda}{3}}x} + c_2^2 e^{-2\sqrt{\frac{\lambda}{3}}x} + 2c_1c_2\right) \Longleftrightarrow \lambda_{F_1} - \frac{4\lambda}{3}c_1c_2 = \frac{4\lambda}{3}c_1c_2 \Longleftrightarrow c_1c_2 = \frac{3\lambda_1}{8\lambda}$. So in this case $f_1(x) = c_1 e^{\sqrt{\frac{\lambda}{3}}x} + c_2 e^{-\sqrt{\frac{\lambda}{3}}x}$, with $c_1, c_2 \in \mathbb{R}$, $c_1c_2 = \frac{3\lambda_{F_1}}{8\lambda}$.

ii) $\lambda = 0 \Longrightarrow f_1(x) = ax + b$. Verifying also the equation (2) we obtain the supplementary condition: $a^2 = -\frac{\lambda_{F_1}}{2}$. So if $\lambda_1 > 0$ then we don't have any solution and if $\lambda_1 \leq 0$ then $f_1(x) = ax + b$, with $a = \sqrt{-\frac{\lambda_1}{2}}$ or $a = -\sqrt{-\frac{\lambda_1}{2}}$, $b \in \mathbb{R}$.

iii)$\lambda < 0 \Rightarrow f_1(x) = c_1 \cos\left(\sqrt{-\frac{\lambda}{3}}x\right) + c_2 \sin\left(\sqrt{-\frac{\lambda}{3}}x\right)$, $c_1, c_2 \in \mathbb{R}$.

Then $f_1'(x) = \sqrt{-\frac{\lambda}{3}}\left[c_2 \cos\left(\sqrt{-\frac{\lambda}{3}}x\right) - c_1 \sin\left(\sqrt{-\frac{\lambda}{3}}x\right)\right]$. Hence λ_{F_1} - $\frac{2\lambda}{3}\left[c_2^2 \cos^2\left(\sqrt{-\frac{\lambda}{3}}x\right) + c_1^2 \sin^2\left(\sqrt{-\frac{\lambda}{3}}x\right) - 2c_1c_2 \cos\left(\sqrt{-\frac{\lambda}{3}}x\right) \sin\left(\sqrt{-\frac{\lambda}{3}}x\right)\right] = \frac{2\lambda}{3}\left[c_1^2 \cos^2\left(\sqrt{-\frac{\lambda}{3}}x\right) + c_2^2 \sin^2\left(\sqrt{-\frac{\lambda}{3}}x\right) + 2c_1c_2 \cos\left(\sqrt{-\frac{\lambda}{3}}x\right) \sin\left(\sqrt{-\frac{\lambda}{3}}x\right)\right]$ $\Longleftrightarrow \lambda_{F_1} = \frac{2\lambda}{3}c_2^2\left[\cos^2\left(\sqrt{-\frac{\lambda}{3}}x\right) + \sin^2\left(\sqrt{-\frac{\lambda}{3}}x\right)\right] + \frac{2\lambda}{3}c_1^2\left[\cos^2\left(\sqrt{-\frac{\lambda}{3}}x\right) + \sin^2\left(\sqrt{-\frac{\lambda}{3}}x\right)\right] \Longleftrightarrow \lambda_{F_1} = \frac{2\lambda}{3}\left(c_1^2 + c_2^2\right) \Longleftrightarrow c_1^2 + c_2^2 = \frac{3\lambda_{F_1}}{2\lambda} > 0$.

So if $\lambda_1 \geq 0$ we don't have any solution and if $\lambda_1 < 0$ then $f_1(x) = c_1 \cos\left(\sqrt{-\frac{\lambda}{3}}x\right) + c_2 \sin\left(\sqrt{-\frac{\lambda}{3}}x\right)$, with $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 = \frac{3\lambda_{F_1}}{2\lambda}$.

V) $M = I \times_{f_1} F_1 \times_{f_2} F_2$ is Ricci flat and $\dim F_1 = k_1 = 1$, $\dim F_2 = k_2 = 2$. Thus, the system (1) becomes:

$$\begin{cases} \frac{f_1''}{f_1} + 2\frac{f_2''}{f_2} = 0 \\ f_1 f_1'' + 2 f_1 f_1' \cdot \frac{f_2'}{f_2} = 0 \\ \lambda_{F_2} + f_2 f_2'' + \left(f_2'\right)^2 + f_2 f_2' \cdot \frac{f_1'}{f_1} = 0 \end{cases} \quad (6)$$

From equations (1) and (2) we obtain that $\frac{f_2''}{f_2} = \frac{f_1'}{f_1} \cdot \frac{f_2'}{f_2}$. This implies that $\left(\frac{f_2'}{f_1}\right)' = \frac{f_2'' f_1 - f_2' f_1'}{f_1^2} = 0 \Longrightarrow f_2' = c f_1,\ c \in \mathbb{R}$.

1) $c = 0 \Longrightarrow f_2(x) = a \in \mathbb{R}$. Thus we obtain $\frac{f_1''}{f_1} = 0 \Longrightarrow f_1'' = 0$. This implies $f_1(x) = \alpha x + \beta,\ f_2(x) = a,\ a \in \mathbb{R},\ \lambda_2 = 0$. If $\lambda_2 \neq 0$ then we don't

have any solution in this case.

2) $c \neq 0 \Longrightarrow f_1 = C_0 f_2',\ C_0 \in \mathbb{R}^*, C_0 = \frac{1}{c}$. Hence equation (3) from the system (6) becomes $\lambda_2 + f_2 f_2'' + \left(f_2'\right)^2 + f_2 f_2' \cdot \frac{C_0 f_2''}{C_0 f_2'} = 0 \Longleftrightarrow \lambda_2 + 2 f_2 f_2'' + \left(f_2'\right)^2 = 0 \quad \left(3'\right)$. Moreover, the equation (1) from the system (6) becomes $\frac{C_0 f_2'''}{C_0 f_2'} + 2 \cdot \frac{f_2''}{f_2} = 0 \Longleftrightarrow \frac{f_2'''}{f_2'} + 2 \cdot \frac{f_2''}{f_2} = 0 \Longleftrightarrow f_2 f_2''' + 2 f_2' f_2'' = 0 \quad \left(3''\right)$. But $\left(f_2^2 f_s''\right)' = f_2 \left(2 f_2' f_2'' + f_2 f_2'''\right) = 0$. Thus $f_2^2 f_2'' = C,\ C \in \mathbb{R}$, which implies $f_2'' f_2 = \frac{C}{f_2}$. At this point, equation $\left(3'\right)$ becomes $\lambda_{F_2} + \left(f_2'\right)^2 + \frac{2C}{f_2} = 0 \Longleftrightarrow \left(f_2'\right)^2 = -\frac{2C}{f_2} - \lambda_{F_2} \left(3'''\right)$.

We give now the solution of the differential equation $\left(3'''\right)$ in the parametrized form.

Let the surface $\mathcal{S} = \left\{(x, f_2, f_2') \in \mathbb{R}^3 \mid \left(f_2'\right)^2 + \frac{2C}{f_2} + \lambda_{F_2} = 0\right\}$ and the parametrization $x = u,\ f_2' = v$ and $f_2 = -\frac{2C}{\lambda_{F_2} + v^2}$.

Then $d\left(-\frac{2C}{\lambda_{F_2} + v^2}\right) = v du \Longleftrightarrow \frac{4C}{\left(\lambda_{F_2} + v^2\right)^2} dv = du \Longleftrightarrow$
$u = x = 4C \int \frac{1}{\left(\lambda_{F_2} + v^2\right)^2} dv$.

i) If $\lambda_{F_2} = 0$ then $x = -\frac{4C}{3v^3},\ f_2(x) = -\frac{2C}{\lambda_{F_2} + v^2}$ and $f_1(x) = C_0 v$, with $C, C_0 \in \mathbb{R},\ C_0 \neq 0$.

ii) If $\lambda_{F_2} > 0$ then $x = 4C \left[\frac{v}{2\lambda_{F_2}\left(v^2 + \lambda_{F_2}\right)} + \frac{1}{2\lambda_{F_2}\sqrt{\lambda_{F_2}}} arctg\left(\frac{v}{\sqrt{\lambda_{F_2}}}\right)\right]$, $f_2(x) = -\frac{2C}{\lambda_{F_2} + v^2}$ and $f_1(x) = C_0 v$, with $C, C_0 \in \mathbb{R},\ C_0 \neq 0$.

iii) If $\lambda_{F_2} < 0$ then $x = 4C \left[\frac{v}{2\lambda_{F_2}\left(v^2 + \lambda_{F_2}\right)} + \frac{1}{4\lambda_{F_2}\sqrt{-\lambda_{F_2}}} \ln \left|\frac{v - \sqrt{-\lambda_{F_2}}}{v + \sqrt{-\lambda_{F_2}}}\right|\right]$, $f_2(x) = -\frac{2C}{\lambda_{F_2} + v^2}$ and $f_1(x) = C_0 v$, with $C, C_0 \in \mathbb{R},\ C_0 \neq 0$.

VI) $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ with $\dim F_1 = \dim F_2 = \dim F_3 = 1$. Thus, the system (1) becomes:

$$\begin{cases} \frac{f_1''}{f_1} + \frac{f_2''}{f_2} + \frac{f_3''}{f_3} = \lambda \\ f_1 f_1'' + f_1 f_1' \left(\frac{f_2'}{f_2} + \frac{f_3'}{f_3} \right) = \lambda f_1^2 \\ f_2 f_2'' + f_2 f_2' \left(\frac{f_1'}{f_1} + \frac{f_3'}{f_3} \right) = \lambda f_2^2 \\ f_3 f_3'' + f_3 f_3' \left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} \right) = \lambda f_3^2 \end{cases} \quad (7)$$

If we consider $A = \frac{f_1'}{f_1}$, $B = \frac{f_2'}{f_2}$, $C = \frac{f_3'}{f_3}$ and $h = A + B + C$ we obtain:

$$\begin{cases} A' + B' + C' + A^2 + B^2 + C^2 = \lambda \\ A' + A(A+B+C) = \lambda \\ B' + B(A+B+C) = \lambda \\ C' + C(A+B+C) = \lambda \end{cases} \quad (8)$$

Summing the last three equations we obtain $h' + h^2 = 3\lambda \Longrightarrow h' = -h^2 + 3\lambda$.

i) $\lambda = 0$. Then $h' = -h^2 \Longrightarrow h(x) = \frac{1}{x+d}$ with $d \in \mathbb{R}$. Thus $A' + A\frac{1}{x+d} = 0 \Longrightarrow A(x) = \frac{a}{x+d}$ with $a \in \mathbb{R}$, $B' + B\frac{1}{x+c} = 0 \Longrightarrow B(x) = \frac{b}{x+d}$ with $b \in \mathbb{R}$ and $C' + C\frac{1}{x+c} = 0 \Longrightarrow C(x) = \frac{c}{x+d}$ with $c \in \mathbb{R}$. Hence $\frac{f_1'}{f_1} = \frac{a}{x+d} \Longrightarrow f_1(x) = a_1(x+d)^a$ with $a_1 \in \mathbb{R}$, $\frac{f_2'}{f_2} = \frac{b}{x+d} \Longrightarrow f_2(x) = b_1(x+d)^b$ with $b_1 \in \mathbb{R}$ and $\frac{f_3'}{f_3} = \frac{c}{x+d} \Longrightarrow f_3(x) = c_1(x+d)^c$ with $c_1 \in \mathbb{R}$. Because $h = A+B+C$ we have that $\frac{1}{x+d} = \frac{a}{x+d} + \frac{b}{x+d} + \frac{c}{x+d} \Longrightarrow a+b+c = 1$. But $A' + B' + C' + A^2 + B^2 + C^2 = 0$ $\Longrightarrow -\frac{a}{(x+d)^2} + \frac{a^2}{(x+d)^2} - \frac{b}{(x+d)^2} + \frac{b^2}{(x+d)^2} - \frac{c}{(x+d)^2} + \frac{c^2}{(x+d)^2} = 0 \Longrightarrow a^2 + b^2 + c^2 = a + b + c = 1$. Thus, the general solution in this case is $f_1(x) = a_1(x+d)^a$, $f_2(x) = b_1(x+d)^b$, $f_3(x) = c_1(x+d)^c$,with $a_1, b_1, c_1 \in \mathbb{R}$, where $a, b, c \in \mathbb{R}$ are satisfying $a^2 + b^2 + c^2 = a + b + c = 1$.

ii) $\lambda < 0$. Then $\frac{h'}{h^2 + \left(\sqrt{-3\lambda}\right)^2} = -1 \Longrightarrow \left[\frac{1}{\sqrt{-3\lambda}} arctg \left(\frac{h}{\sqrt{-3\lambda}} \right) \right]' = -1 \Longrightarrow$ $h(x) = \sqrt{-3\lambda} tg(D - \sqrt{-3\lambda} x)$, $D \in \mathbb{R}$. Thus $A' + hA = \lambda \Longrightarrow A' = -Ah + \lambda$. The reduced equation $\overline{A}' = -\overline{A}h$ has the solution $\overline{A}(x) = \frac{c}{\cos\left(D - \sqrt{-3\lambda}x\right)}$, where $c > 0$. Hence, using the method of constant variables, we obtain the general solution of the form $A = \frac{\frac{\lambda}{\sqrt{-3\lambda}} \sin\left(D - \sqrt{-3\lambda}x\right) + \alpha}{\cos\left(D - \sqrt{-3\lambda}x\right)}$, where $\alpha \in \mathbb{R}$. Similar $B = \frac{\frac{\lambda}{\sqrt{-3\lambda}} \sin\left(D - \sqrt{-3\lambda}x\right) + \beta}{\cos\left(D - \sqrt{-3\lambda}x\right)}$ and $C = \frac{\frac{\lambda}{\sqrt{-3\lambda}} \sin\left(D - \sqrt{-3\lambda}x\right) + \gamma}{\cos\left(D - \sqrt{-3\lambda}x\right)}$, with $\beta, \gamma \in \mathbb{R}$. Since $h =$

$A+B+C$ it results that $\alpha+\beta+\gamma=0$. Also, because $\left(A^{'}+A^{2}\right)+\left(B^{'}+B^{2}\right)+\left(C^{'}+C^{2}\right)=\lambda$, we get that $\alpha^{2}+\beta^{2}+\gamma^{2}=-2\lambda$. But considering the relation $A=\frac{f_{1}^{'}}{f_{1}}=\frac{\frac{\lambda}{\sqrt{-3\lambda}}\sin\left(D-\sqrt{--3\lambda x}\right)+\alpha}{\cos\left(D-\sqrt{-3\alpha x}\right)}=\frac{\lambda}{\sqrt{-3\lambda}}tg\left(D-\sqrt{-3\lambda x}\right)+\frac{\alpha}{\cos\left(D-\sqrt{-3\lambda x}\right)}$ and the other ones for the functions B and C we obtain the following warping functions:

$f_{1}(x)=a_{1}\left[\left|\cos\left(D-\sqrt{-3\lambda}x\right)\right|\right]^{\frac{1}{3}}\left[\left|tg\left(\frac{D-\sqrt{-3\lambda}x}{2}+\frac{\pi}{4}\right)\right|\right]^{\frac{\alpha}{\sqrt{-3\lambda}}}$,

$f_{2}(x)=a_{2}\left[\left|\cos\left(D-\sqrt{-3\lambda}x\right)\right|\right]^{\frac{1}{3}}\left[\left|tg\left(\frac{D-\sqrt{-3\lambda}x}{2}+\frac{\pi}{4}\right)\right|\right]^{\frac{\beta}{\sqrt{-3\lambda}}}$, and

$f_{3}(x)=a_{3}\left[\left|\cos\left(D-\sqrt{-3\lambda}x\right)\right|\right]^{\frac{1}{3}}\left[\left|tg\left(\frac{D-\sqrt{-3\lambda}x}{2}+\frac{\pi}{4}\right)\right|\right]^{\frac{\gamma}{\sqrt{-3\lambda}}}$,

where $a_{1},a_{2},a_{3}>0$, $D,\alpha,\beta,\gamma\in\mathbb{R}$ such that $\alpha+\beta+\gamma=0$ and $\alpha^{2}+\beta^{2}+\gamma^{2}=-2\lambda$.

iii) $\lambda>0$. Then $\frac{h^{'}}{\left(\sqrt{3\lambda}\right)^{2}-h^{2}}=1\Longrightarrow\left[\frac{1}{\sqrt{3\lambda}}arcth\left(\frac{h}{\sqrt{3\lambda}}\right)\right]^{'}=1\Longrightarrow h(x)=\sqrt{3\lambda}\tanh\left(\sqrt{3\lambda}x+D\right)$, where $D\in\mathbb{R}$. Thus, the reduced equation is $\overline{A}^{'}=-\overline{A}h$ has the solution $A=\frac{\alpha}{\cosh\left(\sqrt{3\lambda}x+D\right)}$, where $\alpha>0$. Similar $B=\frac{\beta}{\cosh\left(\sqrt{3\lambda}x+D\right)}$ and $C=\frac{\gamma}{\cosh\left(\sqrt{3\lambda}x+D\right)}$, $\beta,\gamma\in\mathbb{R}$. Hence, using the method of constant variation, we obtain the general solution of the form

$$A(x)=\frac{\frac{\lambda}{\sqrt{3\lambda}}\sinh\left(\sqrt{3\lambda}x+D\right)+\alpha}{\cosh\left(\sqrt{3\lambda}x+D\right)}$$

and similar

$$B(x)=\frac{\frac{\lambda}{\sqrt{3\lambda}}\sinh\left(\sqrt{3\lambda}x+D\right)+\beta}{\cosh\left(\sqrt{3\lambda}x+D\right)}$$

and

$$C(x)=\frac{\frac{\lambda}{\sqrt{3\lambda}}\sinh\left(\sqrt{3\lambda}x+D\right)+\gamma}{\cosh\left(\sqrt{3\lambda}x+D\right)}.$$

Since $h=A+B+C$, it results that $\alpha+\beta+\gamma=0$. Also, because $\left(A^{'}+A^{2}\right)+\left(B^{'}+B^{2}\right)+\left(C^{'}+C^{2}\right)=\lambda$, we get that $\alpha^{2}+\beta^{2}+\gamma^{2}=4\lambda$. But considering the relation of the form $A=\frac{f_{1}^{'}}{f_{1}}=\frac{\frac{\lambda}{\sqrt{3\lambda}}\sinh\left(\sqrt{3\lambda}x+D\right)+\alpha}{\cosh\left(\sqrt{3\lambda}x+D\right)}=\frac{\lambda}{\sqrt{3\lambda}}\tanh\left(\sqrt{3\lambda}x+D\right)+\frac{\alpha}{\cosh\left(\sqrt{3\lambda}x+D\right)}$ and the other ones for the functions B and C we obtain the following warping functions:

$f_{1}(x)=a_{1}\left[\left|\cosh\left(\sqrt{3\lambda}x+D\right)\right|\right]^{\frac{1}{3}}e^{\frac{2\alpha}{\sqrt{3\lambda}}arctg\left(e^{\sqrt{3\lambda}x+D}\right)}$,

$f_2(x) = a_2 \left[\left|\cosh\left(\sqrt{3\lambda}x + D\right)\right|\right]^{\frac{1}{3}} e^{\frac{2\beta}{\sqrt{3\lambda}} arctg\left(e^{\sqrt{3\lambda}x+D}\right)}$,

$f_3(x) = a_3 \left[\left|\cosh\left(\sqrt{3\lambda}x + D\right)\right|\right]^{\frac{1}{3}} e^{\frac{2\gamma}{\sqrt{3\lambda}} arctg\left(e^{\sqrt{3\lambda}x+D}\right)}$, where $a_1, a_2, a_3 > 0$, $D, \alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta + \gamma = 0$ and $\alpha^2 + \beta^2 + \gamma^2 = 4\lambda$.

d) The case when $\dim M = 5$.

VII) $M = I \times_{f_1} F_1$ with $\dim F_1 = k_1 = 4$. Thus, the system (1) becomes:

$$\begin{cases} 4\frac{f_1''}{f_1} = \lambda \\ \lambda_{F_1} + f_1 f_1'' + 3\left(f_1'\right)^2 = \lambda f_1^2 \end{cases} \quad (9)$$

Thus from equation (1) we obtain that $f_1'' = \frac{\lambda}{4} f_1$ and together with equation (2) we obtain $\lambda_1 + 3\left(f_1'\right)^2 = \frac{3\lambda}{4} f_1^2$.

i) $\lambda > 0 \Longrightarrow f_1(x) = c_1 e^{\sqrt{\frac{\lambda}{4}}x} + c_2 e^{-\sqrt{\frac{\lambda}{4}}x}$, $c_1, c_2 \in \mathbb{R}$. Then $f_1'(x) = \sqrt{\frac{\lambda}{4}}\left(c_1 e^{\sqrt{\frac{\lambda}{4}}x} - c_2 e^{-\sqrt{\frac{\lambda}{4}}x}\right)$. Hence $\lambda_{F_1} + \frac{3\lambda}{4}\left(c_1^2 e^{2\sqrt{\frac{\lambda}{4}}x} + c_2^2 e^{-2\sqrt{\frac{\lambda}{4}}x} - 2c_1c_2\right) = \frac{3\lambda}{4}\left(c_1^2 e^{2\sqrt{\frac{\lambda}{4}}x} + c_2^2 e^{-2\sqrt{\frac{\lambda}{4}}x} + 2c_1c_2\right) \Longleftrightarrow \lambda_{F_1} - \frac{6\lambda}{4}c_1c_2 = \frac{6\lambda}{4}c_1c_2 \Longleftrightarrow c_1c_2 = \frac{\lambda_{F_1}}{3\lambda}$. So in this case $f_1(x) = c_1 e^{\sqrt{\frac{\lambda}{4}}x} + c_2 e^{-\sqrt{\frac{\lambda}{4}}x}$, $c_1, c_2 \in \mathbb{R}$, $c_1c_2 = \frac{\lambda_{F_1}}{3\lambda}$.

ii) $\lambda = 0 \Longrightarrow f_1(x) = ax + b$. Verifying also the equation (2) we obtain the supplementary condition: $a^2 = -\frac{\lambda_{F_1}}{3}$. So if $\lambda_1 > 0$ then we don't have any solution and if $\lambda_1 \leq 0$, $f_1(x) = ax + b$, with $a = \sqrt{-\frac{\lambda_1}{3}}$ or $a = -\sqrt{-\frac{\lambda_1}{3}}$, $b \in \mathbb{R}$.

iii)$\lambda < 0 \Longrightarrow f_1(x) = c_1\cos\left(\sqrt{-\frac{\lambda}{4}}x\right) + c_2\sin\left(\sqrt{-\frac{\lambda}{4}}x\right)$, $c_1, c_2 \in \mathbb{R}$.

Then $f_1'(x) = \sqrt{-\frac{\lambda}{4}}\left[c_2\cos\left(\sqrt{-\frac{\lambda}{4}}x\right) - c_1\sin\left(\sqrt{-\frac{\lambda}{4}}x\right)\right]$. Hence $\lambda_{F_1} - \frac{3\lambda}{4}\left[c_2^2\cos^2\left(\sqrt{-\frac{\lambda}{4}}x\right) + c_1^2\sin^2\left(\sqrt{-\frac{\lambda}{4}}x\right) - 2c_1c_2\cos\left(\sqrt{-\frac{\lambda}{4}}x\right)\sin\left(\sqrt{-\frac{\lambda}{4}}x\right)\right] = \frac{3\lambda}{4}\left[c_1^2\cos^2\left(\sqrt{-\frac{\lambda}{4}}x\right) + c_2^2\sin^2\left(\sqrt{-\frac{\lambda}{4}}x\right) + 2c_1c_2\cos\left(\sqrt{-\frac{\lambda}{4}}x\right)\sin\left(\sqrt{-\frac{\lambda}{4}}x\right)\right]$ $\Longleftrightarrow \lambda_{F_1} = \frac{3\lambda}{4}c_2^2\left[\cos^2\left(\sqrt{-\frac{\lambda}{4}}x\right) + \sin^2\left(\sqrt{-\frac{\lambda}{4}}x\right)\right] + \frac{3\lambda}{4}c_1^2\left[\cos^2\left(\sqrt{-\frac{\lambda}{4}}x\right) + \sin^2\left(\sqrt{-\frac{\lambda}{4}}x\right)\right] \Longleftrightarrow \lambda_{F_1} = \frac{3\lambda}{4}\left(c_1^2 + c_2^2\right) \Longleftrightarrow c_1^2 + c_2^2 = \frac{4\lambda_{F_1}}{3\lambda} > 0$. So if $\lambda_1 \geq 0$ we don't have any solution and if $\lambda_1 < 0$ then $f_1(x) = c_1\cos\left(\sqrt{-\frac{\lambda}{4}}x\right) + c_2\sin\left(\sqrt{-\frac{\lambda}{4}}x\right)$, with $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 = \frac{4\lambda_{F_1}}{3\lambda}$.

VIII) $M = I \times_{f_1} F_1 \times_{f_2} F_2$ with $\dim F_1 = 1$, $\dim F_2 = k = 3$. Thus, the system (1) becomes:

$$\left\{ \begin{array}{l} \frac{f_1^{''}}{f_1} + 3\frac{f_2^{''}}{f_2} = \lambda \\ \frac{f_1^{''}}{f_1} + 3\frac{f_1^{'}}{f_1} \cdot \frac{f_2^{'}}{f_2} = \lambda \\ \lambda_{F_2} + f_2 f_2^{''} + 2\left(f_2^{'}\right)^2 + \frac{f_2 f_2^{'} f_1^{'}}{f_1} = \lambda \end{array} \right. \tag{10}$$

From (1) and (2) we obtain that $\frac{f_2^{''}}{f_2} = \frac{f_1^{'} f_2^{'}}{f_1 f_2}$, which involves that $f_1 = c_0 f_2^{'}$, $c_0 \in \mathbb{R}$ and $f_2 f_2^{''} = \frac{f_2 f_1^{'} f_2^{'}}{f_1}$. Thus equation (3) becomes $\lambda_{F_2} + 2 f_2 f_2^{''} + 2\left(f_2^{'}\right)^2 = \lambda$. This implies that $2\left(f_2 f_2^{''} + \left(f_2^{'}\right)^2\right) = \lambda - \lambda_{F_2} \Longrightarrow f_2 f_2^{''} + \left(f_2^{'}\right)^2 = \frac{\lambda - \lambda_{F_2}}{2} \Longrightarrow \left(f_2 f_2^{'}\right)^{'} = f_2 f_2^{''} + \left(f_2^{'}\right)^2 = \frac{\lambda-\lambda_{F_2}}{2} \Longrightarrow f_2 f_2^{'} = \frac{\lambda-\lambda_{F_2}}{2} x + A \Longrightarrow \left(f_2^2\right)^{'} = 2 f_2 f_2^{'} = (\lambda - \lambda_{F_2})x + 2A \Longrightarrow f_2^2 = \frac{\lambda-\lambda_{F_2}}{2} x^2 + 2Ax + B$. Hence $f_2(x) = \sqrt{\frac{\lambda-\lambda_{F_2}}{2} x^2 + 2Ax + B}$ and $f_1(x) = \frac{c_0[(\lambda-\lambda_{F_2})x + 2A]}{2\sqrt{\frac{\lambda-\lambda_{F_2}}{2} x^2 + 2Ax + B}}$, where $c_0, A, B \in \mathbb{R}$ such that $\frac{\lambda-\lambda_{F_2}}{2} x^2 + 2Ax + B > 0$ and $c_0[(\lambda - \lambda_{F_2})x + 2A] > 0$ for every $x \in I$.

IX) $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3 \times_{f_4} F_4$ with $\dim F_1 = \dim F_2 = \dim F_3 = \dim F_4 = 1$. Thus, the system (1) becomes:

$$\left\{ \begin{array}{l} \frac{f_1^{''}}{f_1} + \frac{f_2^{''}}{f_2} + \frac{f_3^{''}}{f_3} + \frac{f_4^{''}}{f_4} = \lambda \\ f_1 f_1^{''} + f_1 f_1^{'}\left(\frac{f_2^{'}}{f_2} + \frac{f_3^{'}}{f_3} + \frac{f_4^{'}}{f_4}\right) = \lambda f_1^2 \\ f_2 f_2^{''} + f_2 f_2^{'}\left(\frac{f_1^{'}}{f_1} + \frac{f_3^{'}}{f_3} + \frac{f_4^{'}}{f_4}\right) = \lambda f_2^2 \\ f_3 f_3^{''} + f_3 f_3^{'}\left(\frac{f_1^{'}}{f_1} + \frac{f_2^{'}}{f_2} + \frac{f_4^{'}}{f_4}\right) = \lambda f_3^2 \\ f_4 f_4^{''} + f_4 f_4^{'}\left(\frac{f_1^{'}}{f_1} + \frac{f_2^{'}}{f_2} + \frac{f_3^{'}}{f_3}\right) = \lambda f_4^2 \end{array} \right. \tag{11}$$

If we consider $A = \frac{f_1^{'}}{f_1}$, $B = \frac{f_2^{'}}{f_2}$, $C = \frac{f_3^{'}}{f_3}$, $D = \frac{f_4^{'}}{f_4}$ and $h = A + B + C + D$ we obtain:

$$\left\{ \begin{array}{l} A^{'} + B^{'} + C^{'} + D^{'} + A^2 + B^2 + C^2 + D^2 = \lambda \\ A^{'} + Ah = \lambda \\ B^{'} + Bh = \lambda \\ C^{'} + Ch = \lambda \\ D^{'} + Dh = \lambda \end{array} \right. \tag{12}$$

Summing the last four equations we obtain $h^{'} + h^2 = 4\lambda \Longrightarrow h^{'} = -h^2 + 4\lambda$.

i) $\lambda = 0$. Then $h^{'} = -h^2 \Longrightarrow h(x) = \frac{1}{x+e}$ with $e \in \mathbb{R}$. Thus $A^{'} + A\frac{1}{x+e} = 0 \Longrightarrow A(x) = \frac{a}{x+e}$ with $a \in \mathbb{R}$, $B^{'} + B\frac{1}{x+e} = 0 \Longrightarrow B(x) = \frac{b}{x+e}$ with $b \in \mathbb{R}$, $C^{'} + C\frac{1}{x+e} = 0 \Longrightarrow C(x) = \frac{c}{x+e}$ with $c \in \mathbb{R}$ and $D^{'} + D\frac{1}{x+e} = 0 \Longrightarrow D(x) = \frac{d}{x+e}$ with $d \in \mathbb{R}$.. Hence $\frac{f_1^{'}}{f_1} = \frac{a}{x+e} \Longrightarrow f_1(x) = a_1(x+e)^a$ with $a_1 \in \mathbb{R}$, $\frac{f_2^{'}}{f_2} = \frac{b}{x+e} \Longrightarrow f_2(x) = b_1(x+e)^b$ with $b_1 \in \mathbb{R}$, $\frac{f_3^{'}}{f_3} = \frac{c}{x+e} \Longrightarrow f_3(x) = c_1(x+e)^c$ with $c_1 \in \mathbb{R}$ and $\frac{f_4^{'}}{f_4} = \frac{d}{x+e} \Longrightarrow f_4(x) = c_1(x+e)^d$ with $c_1 \in \mathbb{R}$. Because $h = A+B+C+D$ we have that $\frac{1}{x+e} = \frac{a}{x+e} + \frac{b}{x+e} + \frac{c}{x+e} + \frac{d}{x+e} \Longrightarrow a+b+c+d = 1$. But $A^{'} + B^{'} + C^{'} + D^{'} + A^2 + B^2 + C^2 + D^2 = 0 \Longrightarrow -\frac{a}{(x+e)^2} + \frac{a^2}{(x+e)^2} - \frac{b}{(x+e)^2} + \frac{b^2}{(x+e)^2} - \frac{c}{(x+e)^2} + \frac{c^2}{(x+e)^2} - \frac{d}{(x+e)^2} + \frac{d^2}{(x+e)^2} = 0 \Longrightarrow a^2+b^2+c^2+d^2 = a+b+c+d = 1$. Thus the general solution in this case is $f_1(x) = a_1(x+e)^a$, $f_2(x) = b_1(x+e)^b$, $f_3(x) = c_1(x+e)^c$, $f_4(x) = d_1(x+e)^d$, with $a_1, b_1, c_1, d_1 \in \mathbb{R}$, where $a, b, c, d \in \mathbb{R}$ are satisfying $a^2+b^2+c^2+d^2 = a+b+c+d = 1$.

ii) $\lambda < 0$. Then $\frac{h^{'}}{h^2+\left(\sqrt{-4\lambda}\right)^2} = -1 \Longrightarrow \left[\frac{1}{\sqrt{-4\lambda}} arctg\left(\frac{h}{\sqrt{-4\lambda}}\right)\right]^{'} = -1 \Longrightarrow h(x) = \sqrt{-4\lambda}tg(E - \sqrt{-4\lambda}x)$, $E \in \mathbb{R}$. Thus $A^{'} + hA = \lambda \Longrightarrow A^{'} = -Ah + \lambda$. The reduced equation $\overline{A}^{'} = -\overline{A}h$ has the solution $\overline{A}(x) = \frac{c}{\cos\left(E-\sqrt{-4\lambda}x\right)}$, where $c > 0$. Hence, using the method of constant variation, we obtain the general

solution of the form $A = \frac{\frac{\lambda}{\sqrt{-4\lambda}}\sin\left(E-\sqrt{-4\lambda}x\right)+\alpha}{\cos\left(E-\sqrt{-4\lambda}x\right)}$, where $\alpha \in \mathbb{R}$. Similar $B = \frac{\frac{\lambda}{\sqrt{-4\lambda}}\sin\left(E-\sqrt{-4\lambda}x\right)+\beta}{\cos\left(E-\sqrt{-4\lambda}x\right)}$, $C = \frac{\frac{\lambda}{\sqrt{-4\lambda}}\sin\left(E-\sqrt{-4\lambda}x\right)+\gamma}{\cos\left(E-\sqrt{-4\lambda}x\right)}$ and $D = \frac{\frac{\lambda}{\sqrt{-4\lambda}}\sin\left(E-\sqrt{-4\lambda}x\right)+\delta}{\cos\left(E-\sqrt{-4\lambda}x\right)}$, with $\beta, \gamma, \delta \in \mathbb{R}$. Since $h = A+B+C+D$ it results that $\alpha+\beta+\gamma+\delta = 0$. Also, because $\left(A^{'} + A^2\right) + \left(B^{'} + B^2\right) + \left(C^{'} + C^2\right) + \left(D^{'} + D^2\right) = \lambda$, we get that

$\alpha^2+\beta^2+\gamma^2+\delta^2 = -3\lambda$. But considering the relation of the form $A = \frac{f_1^{'}}{f_1} = \frac{\frac{\lambda}{\sqrt{-4\lambda}}\sin\left(E-\sqrt{--4\lambda}x\right)+\alpha}{\cos\left(E-\sqrt{-4\alpha}x\right)} = \frac{\lambda}{\sqrt{-4\lambda}}tg\left(E - \sqrt{-4\lambda}x\right) + \frac{\alpha}{\cos\left(E-\sqrt{-4\lambda}x\right)}$ and the other ones for the functions B, C and D we obtain the following warping functions:

$f_1(x) = a_1\left[\left|\cos\left(E - \sqrt{-4\lambda}x\right)\right|\right]^{\frac{1}{4}}\left[\left|tg\left(\frac{E-\sqrt{-4\lambda}x}{2} + \frac{\pi}{4}\right)\right|\right]^{\frac{\alpha}{\sqrt{-4\lambda}}}$,

$f_2(x) = a_2\left[\left|\cos\left(E - \sqrt{-4\lambda}x\right)\right|\right]^{\frac{1}{4}}\left[\left|tg\left(\frac{E-\sqrt{-4\lambda}x}{2} + \frac{\pi}{4}\right)\right|\right]^{\frac{\beta}{\sqrt{-4\lambda}}}$,

$f_3(x) = a_3\left[\left|\cos\left(E - \sqrt{-4\lambda}x\right)\right|\right]^{\frac{1}{4}}\left[\left|tg\left(\frac{E-\sqrt{-4\lambda}x}{2} + \frac{\pi}{4}\right)\right|\right]^{\frac{\gamma}{\sqrt{-4\lambda}}}$,

$f_4(x) = a_4\left[\left|\cos\left(E - \sqrt{-4\lambda}x\right)\right|\right]^{\frac{1}{4}}\left[\left|tg\left(\frac{E-\sqrt{-4\lambda}x}{2} + \frac{\pi}{4}\right)\right|\right]^{\frac{\delta}{\sqrt{-4\lambda}}}$, where a_1, a_2, a_3, $a_4 > 0$, $E, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha+\beta+\gamma+\delta = 0$ and $\alpha^2+\beta^2+\gamma^2+\delta^2 = -3\lambda$.

iii) $\lambda > 0$. Then $\frac{h^{'}}{\left(\sqrt{4\lambda}\right)^2 - h^2} = 1 \Longrightarrow \left[\frac{1}{\sqrt{4\lambda}} arcth\left(\frac{h}{\sqrt{4\lambda}}\right)\right]^{'} = 1 \Longrightarrow h(x) = \sqrt{4\lambda}\tanh\left(\sqrt{4\lambda}x + E\right)$, where $E \in \mathbb{R}$. Thus, the reduced equation is $\overline{A}^{'} = -\overline{A}h$ has the solution $A = \frac{\alpha}{\cosh\left(\sqrt{4\lambda}x+E\right)}$, where $\alpha > 0$. Similar $B = \frac{\beta}{\cosh\left(\sqrt{4\lambda}x+E\right)}$, $C = \frac{\gamma}{\cosh\left(\sqrt{4\lambda}x+E\right)}$ and $D = \frac{\delta}{\cosh\left(\sqrt{4\lambda}x+E\right)}$, $\beta, \gamma, \delta \in \mathbb{R}$. Hence, using the method of constant variantion, we obtain the general solution of the form $A(x) = \frac{\frac{\lambda}{\sqrt{4\lambda}}\sinh\left(\sqrt{4\lambda}x+E\right)+\alpha}{\cosh\left(\sqrt{4\lambda}x+E\right)}$ and similar $B(x) = \frac{\frac{\lambda}{\sqrt{4\lambda}}\sinh\left(\sqrt{4\lambda}x+E\right)+\beta}{\cosh\left(\sqrt{4\lambda}x+E\right)}$, $C(x) = \frac{\frac{\lambda}{\sqrt{4\lambda}}\sinh\left(\sqrt{4\lambda}x+E\right)+\gamma}{\cosh\left(\sqrt{4\lambda}x+e\right)}$ and $D(x) = \frac{\frac{\lambda}{\sqrt{4\lambda}}\sinh\left(\sqrt{4\lambda}x+E\right)+\delta}{\cosh\left(\sqrt{4\lambda}x+E\right)}$. Since $h = A + B + C + D$, it results that $\alpha + \beta + \gamma + \delta = 0$. Also, because $\left(A^{'} + A^2\right) + \left(B^{'} + B^2\right) + \left(C^{'} + C^2\right) + \left(D^{'} + D^2\right) = \lambda$, we get that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5\lambda$. But considering the relation of the form $A = \frac{f_1^{'}}{f_1} = \frac{\frac{\lambda}{\sqrt{4\lambda}}\sinh\left(\sqrt{4\lambda}x+E\right)+\alpha}{\cosh\left(\sqrt{4\lambda}x+E\right)} = \frac{\lambda}{\sqrt{4\lambda}}\tanh\left(\sqrt{4\lambda}x + E\right) + \frac{\alpha}{\cosh\left(\sqrt{4\lambda}x+E\right)}$ and the other ones for the functions B, C and D we obtain the following warping functions:

$f_1(x) = a_1\left[\left|\cosh\left(\sqrt{4\lambda}x + E\right)\right|\right]^{\frac{1}{4}} e^{\frac{2\alpha}{\sqrt{4\lambda}} arctg\left(e^{\sqrt{4\lambda}x+E}\right)}$,

$f_2(x) = a_2\left[\left|\cosh\left(\sqrt{4\lambda}x + E\right)\right|\right]^{\frac{1}{4}} e^{\frac{2\beta}{\sqrt{4\lambda}} arctg\left(e^{\sqrt{4\lambda}x+E}\right)}$,

$f_3(x) = a_3\left[\left|\cosh\left(\sqrt{4\lambda}x + E\right)\right|\right]^{\frac{1}{4}} e^{\frac{2\gamma}{\sqrt{4\lambda}} arctg\left(e^{\sqrt{4\lambda}x+E}\right)}$,

$f_4(x) = a_4\left[\left|\cosh\left(\sqrt{4\lambda}x + E\right)\right|\right]^{\frac{1}{4}} e^{\frac{2\delta}{\sqrt{4\lambda}} arctg\left(e^{\sqrt{4\lambda}x+E}\right)}$, where $a_1, a_2, a_3, a_4 > 0$, $E, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta = 0$ and $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 5\lambda$.

e) The case when $\dim M \geq 6$.

X) $M = I \times_{f_1} F_1 \times ... \times_{f_m} F_m$ with $\dim F_i = 1$ for every $i \in 1, ..., m\}$. Then $\lambda_{F_i} = 0$ for every $i \in 1, ..., m\}$. In this case, the system (1) becomes:

$$\begin{cases} \text{Each fiber } (F_i, g_{F_i}) \text{ is Einstein with Ricci curvature } \lambda_{F_i} = 0, \\ \sum\limits_{i=1}^{m} \frac{f_i^{''}}{f_i} = \lambda, \\ f_i f_i^{''} + f_i f_i^{'} \sum\limits_{j=1, j\neq i}^{m} \frac{f_j^{'}}{f_j} = \lambda f_i^2 \text{ for every } i \in \{1, ..., m\}. \end{cases} \tag{13}$$

Equation (3) is equivalent to $\frac{f_i^{''}}{f_i} - \left(\frac{f_i^{'}}{f_i}\right)^2 + \frac{f_i^{'}}{f_i}\sum\limits_{j=1}^{m}\frac{f_j^{'}}{f_j} = \lambda$ for every $i \in \{1, ..., m\}$. We consider now the functions $A_i = \frac{f_i^{'}}{f_i}$ for every $i \in \{1, ..., m\}$ and $h = \sum\limits_{i=1}^{m} A_i$. Hence, putting this in (13) we obtain:

$$\begin{cases} \sum_{i=1}^{m}\left(A_i^{'}+A_i^2\right)=\lambda, \\ A_i^{'}+A_ih=\lambda \text{ for every } i\in\{1,...,m\} \end{cases} \tag{14}$$

Summing after i the last m relations of equation (2) we get that: $h'+h^2=\lambda m \Longrightarrow h^{'}=-h^2+A$, where $A=\lambda m$.

i) $A=0 \Longleftrightarrow \lambda=0$. Then we obtain $h'+h^2=0 \Longrightarrow h(t)=\frac{1}{t+C} \Longrightarrow A_i^{'}+\frac{1}{t+C}A_i=0 \Longrightarrow A_i(t)=\frac{c_i}{t+C}$ where $C\in\mathbb{R}$, $c_i\in\mathbb{R}$ for every $i\in\{1,...,m\}$. But $h=\sum_{i=1}^{m}A_i$ which implies $\frac{1}{t+C}=\sum_{i=1}^{m}\frac{c_i}{t+C}$ and thus $\sum_{i=1}^{m}c_i=1$. Because $\sum_{i=1}^{m}\left(A_i^{'}+A_i^2\right)=0$ we have $\sum_{i=1}^{m}\frac{c_i^2-c_i}{(t+C)^2}=0$, which implies $\sum_{i=1}^{m}c_i^2=\sum_{i=1}^{m}c_i$. Finally $A_i=\frac{f_i^{'}}{f_i}=\frac{c_i}{t+C}$ implies $f_i(t)=a_i\,(t+C)^{c_i}$ for every $i\in\{1,...,m\}$. Hence we obtain the following warping functions: $f_i(t)=a_i(t+C)^{c_i}$, where $a_i,c_i,C\in\mathbb{R}$ for every $i\in\{1,...,m\}$, with $\sum_{i=1}^{m}c_i^2=\sum_{i=1}^{m}c_i=1$.

ii) $A<0 \Longleftrightarrow \lambda<0$. Then we put $A=-c$, $c\in(0,\infty)$ and we obtain$\frac{h^{'}}{h^2+c}=-1 \Longrightarrow \left(\frac{1}{\sqrt{c}}Arctg\left(\frac{h}{\sqrt{c}}\right)\right)^{'}=-1 \Longrightarrow h(t)=\sqrt{c}tg(-\sqrt{c}t+K) \Longrightarrow h(t)=\sqrt{-A}tg(K-\sqrt{-A}t)$, $K\in\mathbb{R}$. Thus, the reduced equation is $A_i^{'}+hA_i=0 \Longrightarrow \frac{A_i^{'}}{A_i}=-h=\sqrt{-A}tg(\sqrt{-A}t-K) \Longrightarrow A_i=\frac{c_i}{\cos(\sqrt{-A}t-K)}$, where $c_i>0$ for every $i\in\{1,...,m\}$. Hence, using the method of constant variation, we obtain the general solution $A_i^{'}+hA_i=\lambda \Longrightarrow A_i=\frac{\frac{\lambda}{\sqrt{-A}}\sin\left(\sqrt{-A}t-K\right)+d_i}{\cos\left(\sqrt{-A}t-K\right)}$, where $d_i\in\mathbb{R}$ for every $i\in\{1,...,m\}$. Since $h=\sum_{i=1}^{m}A_i$ it results that $\sum_{i=1}^{m}d_i=0$. Also, bec,ause $\sum_{i=1}^{m}\left(A_i^{'}+A_i^2\right)=\lambda$, we get that $\sum_{i=1}^{m}d_i^2=\lambda\,(1-m)$. But $A_i=\frac{f_i^{'}}{f_i}=\frac{\frac{\lambda}{\sqrt{-A}}\sin\left(\sqrt{-A}t-K\right)+d_i}{\cos\left(\sqrt{-A}t-K\right)}=\frac{\lambda}{\sqrt{-A}}tg\left(\sqrt{-A}t-K\right)+\frac{d_i}{\cos\left(\sqrt{-A}t-K\right)}$ implies that the warping functions are of the form $f_i(t)=a_i\left[\left|\cos\left(\sqrt{-A}t-K\right)\right|\right]^{\frac{\lambda}{A}}\left[\left|tg\left(\frac{\sqrt{-A}t-K}{2}+\frac{\pi}{4}\right)\right|\right]^{\frac{d_i}{\sqrt{-A}}}$, where $a_i>0, d_i\in\mathbb{R}$ for every $i\in\{1,...,m\}$, with $\sum_{i=1}^{m}d_i=0$ and $\sum_{i=1}^{m}d_i^2=\lambda\,(1-m)$.

iii) $A>0 \Longleftrightarrow \lambda>0$. Then $\frac{h^{'}}{A-h^2}=1 \Longrightarrow \left(\frac{1}{\sqrt{A}}arcth\left(\frac{h}{\sqrt{A}}\right)\right)^{'}=1 \Longrightarrow h(t)=\sqrt{A}\tanh\left(\sqrt{A}t+K\right)$ where $K\in\mathbb{R}$. Thus, the reduced equation is $A_i^{'}+hA_i=0 \Longrightarrow \frac{A_i^{'}}{A_i}=-h=-\sqrt{A}\tanh\left(\sqrt{A}t+K\right) \Longrightarrow A_i=\frac{c_i}{\cosh\left(\sqrt{A}t+K\right)}$, where $c_i>0$ for every $i\in\{1,...,m\}$. Hence, using the method of constant variation, we obtain the general solution of the from $A_i^{'}+hA_i=$

$\lambda \Longrightarrow A_i(t) = \frac{\frac{\lambda}{\sqrt{A}}\sinh\left(\sqrt{A}t+K\right)+d_i}{\cosh\left(\sqrt{A}t+K\right)}$, where $d_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$. Since $h = \sum_{i=1}^{m} A_i$, it results that $\sum_{i=1}^{m} d_i = 0$. Also, because $\sum_{i=1}^{m}\left(A_i^{'} + A_i^2\right) = \lambda$, we get that $\sum_{i=1}^{m} d_i^2 = \lambda\left(1+m\right)$. But $A_i = \frac{f_i^{'}}{f_i} = \frac{\frac{\lambda}{\sqrt{A}}\sinh\left(\sqrt{A}t+K\right)+d_i}{\cosh\left(\sqrt{A}t+K\right)} = \frac{\lambda}{\sqrt{A}}\tanh\left(\sqrt{A}t+K\right) + \frac{d_i}{\cosh\left(\sqrt{A}t+K\right)}$ implies that the warping functions are $f_i(t) = a_i\left[\left|\cosh\left(\sqrt{A}t+K\right)\right|\right]^{\frac{\lambda}{A}} e^{\frac{2d_i}{\sqrt{A}}Arctg\left(e^{\sqrt{A}t+K}\right)}$, where $a_i > 0, d_i \in \mathbb{R}$ for every $i \in \{1,...,m\}$, with $\sum_{i=1}^{m} d_i = 0$ and $\sum_{i=1}^{m} d_i^2 = \lambda\left(1+m\right)$.

XI) $M = I \times_{f_1} F_1 \times_{f_2} F_2$ with $\dim F_1 = 1$, $\dim F_2 = k \geq 4$ and M Ricci flat. Hence the system (1) becomes:

$$\begin{cases} \frac{f_1^{''}}{f_1} + k\frac{f_2^{''}}{f_2} = 0 \\ \frac{f_1^{''}}{f_1} + k\frac{f_1^{'}}{f_1}\cdot\frac{f_2^{'}}{f_2} = 0 \\ \lambda_{F_2} + f_2 f_2^{''} + (k-1)\left(f_2^{'}\right)^2 + \frac{f_2 f_2^{'} f_1^{'}}{f_1} = 0 \end{cases} \tag{15}$$

From equations (1) and (2) we obtain that $\frac{f_2^{''}}{f_2} = \frac{f_1^{'}f_2^{'}}{f_1 f_2}$, which involves that $f_1 = c_0 f_2^{'}$, $c_0 \in \mathbb{R}$ and $f_2 f_2^{''} = \frac{f_2 f_1^{'} f_2^{'}}{f_1}$. Thus equation (3) becomes $\lambda_{F_2} + 2f_2 f_2^{''} + (k-1)\left(f_2^{'}\right)^2 = 0$. Also equation (1) becomes $f_2^{'''}f_2 + kf_2^{'}f_2^{''} = 0$. We remark that $\left(f_2^k f_2^{''}\right)^{'} = f_2^{k-1}\left(kf_2^{'}f_2^{''} + f_2^{'''}f_2\right) = 0$ and so $f_2^k f_2^{''} = a \in \mathbb{R}$ which implies $f_2 f_2^{''} = \frac{a}{f_2^{k-1}}$. Hence equation (3) becomes $\lambda_{F_2} + \frac{2a}{f_2^{k-1}} + (k-1)\left(f_2^{'}\right)^2 = 0 \Longrightarrow \left(f_2^{'}\right)^2 = \frac{A}{f_2^{k-1}} + B$, where $A = -\frac{2a}{k-1}$ and $B = -\frac{\lambda_{F_2}}{k-1}$.

Rewriting the above equation in the parametrized form we have the following solution: let the surface $\mathcal{S} = \left\{\left(x, f_2, f_2^{'}\right) \in \mathbb{R}^3 \mid \left(f_2^{'}\right)^2 - \frac{A}{f_2^{k-1}} - B = 0\right\}$ and the parametrization $x = u$, $f_2^{'} = v$ and $f_2 = \sqrt[k-1]{\frac{A}{v^2-B}}$.

Thus $d\left(\sqrt{k-1}\frac{A}{v^2-B}\right) = vdu \Longleftrightarrow \frac{Cv}{\sqrt{k-1}(v^2-B)^k} = vdu \Longleftrightarrow$
$u = C\int\frac{1}{\sqrt{k-1}(v^2-B)^k}dv$ where $C = -\frac{2a}{(k-1)\cdot\sqrt{k-1}A^{k-2}} \in \mathbb{R}$.

Hence the solution in the parametrized form is $x = u = C\int\frac{1}{\sqrt[k-1]{(v^2-B)^k}}dv$, $f_2 = \sqrt{k-1}\frac{A}{v^2-B}$ and $f_1 = c_0 v$, where $c_0, C, A, B \in \mathbb{R}$, $A = -\frac{2a}{k-1}$, $B = -\frac{\lambda_{F_2}}{k-1}$ and $C = -\frac{2a}{(k-1)\cdot\sqrt[k-1]{A^{k-2}}}$.

f) Equal warping functions

XII) $M = I \times_{f_1} F_1 \times ... \times_{f_m} F_m$, where $\dim F_i = k_i \geq 1$ such that $f_i = f$ for every $i \in \{1,...,m\}$, $m \geq 2$. Then the system (1) becomes:

$$\begin{cases} \text{Each fiber } (F_i, g_{F_i}) \text{ is Einstein with Ricci curvature } \lambda_{F_i}, \\ \sum\limits_{i=1}^{m} k_i \frac{f''}{f} = \lambda, \\ \lambda_{F_i} + ff'' + (k_i - 1)(f')^2 + ff' \sum\limits_{j=1, j\neq i}^{m} k_j \frac{f'}{f} = \lambda f^2, \\ \text{ for every } i \in \{1, ..., m\} \end{cases} \tag{16}$$

Thus we obtain $\lambda_{F_i} = a$ for every $i \in \{1, ..., m\}$ and the system becomes:

$$\begin{cases} \textit{Each fiber } (F_i, g_{F_i}) \textit{ is Einstein with Ricci curvature } a \\ f'' = \frac{\lambda}{A} f \\ a + ff'' + (A-1)(f')^2 = \lambda f^2 \end{cases} \tag{17}$$

where $a = \lambda_{F_i}$ for every $i \in \{1, ..., m\}$ and $A = \sum\limits_{i=1}^{m} k_i \geq m$. Hence we have the following situations:

a) If $\lambda > 0$, then $f(x) = c_1 e^{-\sqrt{\frac{\lambda}{A}}x} + c_2 e^{\sqrt{\frac{\lambda}{A}}x}$, $c_1, c_2 \in \mathbb{R}$, with $a = \frac{4(A-1)\lambda}{A} c_1 c_2$ and $A = \sum\limits_{i=1}^{m} k_i$.

b) If $\lambda = 0$, then $f(x) = c_1 + c_2 x$, $c_1, c_2 \in \mathbb{R}$, with $a = -\,(A-1)\,c_2^2$ and $A = \sum\limits_{i=1}^{m} k_i$.

c) If $\lambda < 0$, then $f(x) = c_1 \cos\left(\sqrt{-\frac{\lambda}{A}}x\right) + c_2 \sin\left(\sqrt{-\frac{\lambda}{A}}x\right)$, $c_1, c_2 \in \mathbb{R}$, with $a = \frac{(A-1)\lambda}{A}\left(c_1^2 + c_2^2\right)$ and $A = \sum\limits_{i=1}^{m} k_i$.

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