

A Remark on Linear Codes

Dan Dumitru

Faculty of Mathematics and Informatics

Spiru Haret University

Bucharest, Romania

dandumitru1984@yahoo.com

Abstract

The aim of this paper is to give an obstruction to the existence of $[n, k, d]$ -linear codes. We prove in particular that there are no $[2k + 1, k, k + 1]$ -binary linear codes for every $k \geq 4$. For the ternary case we prove in particular that there are no $[3k + 1, k, 2k + 1]$ -ternary linear codes for every $k \geq 3$. We also prove that there is no linear code C over \mathbb{F}_q of parameters $[qk, k, (q - 1)k + 1]$ for every $k \geq 2$.

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1. Introduction

The study of the existence of certain linear codes has been a main topic in the code theory for many years and bounds to linear or non-linear codes have been given during the time ([1], [2], [3]). In this article we give some results about the non-existence of linear codes starting from the well-known Griesmer and Singleton's bounds. We will first consider the following notations: \mathbb{F}_q is the finite field of q elements and C is a linear code of parameters $[n, k, d]$, which means the codewords of C have length n , C has dimension k and the minimum distance is $d = \min_{c \in C} wt(c)$, where by $wt(c)$ we have denoted the Hamming weight of a codeword $c \in C$.

The following two theorems about the existence of a code are well-known.

Theorem 1. ([2], [3], Griesmer's bound) *Let C be a linear code over \mathbb{F}_q of parameters $[n, k, d]$. Then $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil$, where $\lceil x \rceil$ is the smallest integer greater than x .*

Theorem 2. ([1], Singleton's bound) *For $q, n, d \in \mathbb{N}$, $q \geq 2$ it holds that $A(n, d) \leq q^{n-d+1}$, where $A(n, d) = \max \{M \in \mathbb{N} \mid \text{an } [n, M, d] \text{ code exists}\}$. In particular for a linear code C over \mathbb{F}_q of parameters $[n, k, d]$, we have $k + d \leq n + 1$.*

2. Main results

We can state now the main results using the above two theorems.

Theorem 3. *Let the parameters $[n, k, d]$ be such that $n, k, d \geq 1$ and $q \geq 2$.*

- (i) *Suppose $q = 2$ and $n = 2k + 1$. Then there is no linear code C over \mathbb{F}_q if:*

$$\begin{cases} d \geq k + 3, & \text{for every } k \geq 1 \\ d = k + 2, & \text{for every } k \geq 2 \\ d = k + 1, & \text{for every } k \geq 4 \end{cases}$$

- (ii) *Suppose $q \geq 3$ and $n = qk + 1$. Then there is no linear code C over \mathbb{F}_q if:*

$$\begin{cases} d \geq (q - 1)k + 3, & \text{for every } k \geq 1 \\ d = (q - 1)k + 2, & \text{for every } k \geq 2 \\ d = (q - 1)k + 1, & \text{for every } k \geq 3 \end{cases}$$

Proof:

- (i) Let $q = 2$ and $n = 2k + 1$.

- a) If $d \geq k + 3$, then $k + d \geq 2k + 3 > 2k + 2 = n + 1$ and thus from Singleton's bound there is no linear code C .
- b) If $d = k + 2$, then:

$$\begin{aligned} \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil &= \sum_{i=0}^{k-1} \left\lceil \frac{k+2}{2^i} \right\rceil \geq \sum_{i=0}^{k-1} \frac{k+2}{2^i} = (k+2) \sum_{i=0}^{k-1} \frac{1}{2^i} = \\ &= (k+2) \cdot \frac{2^k - 1}{2^{k-1}} > 2k + 1 \iff \\ \iff (k+2)(2^k - 1) &> (2k+1)2^{k-1} \iff 2^k > 2^{k-1} + k + 2 \end{aligned}$$

inequality which is true by induction for every $k \geq 2$ and thus from Griesmer's bound there is no linear code C .

- c) If $d = k + 1$, then:

$$\begin{aligned} \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil &= \sum_{i=0}^{k-1} \left\lceil \frac{k+1}{2^i} \right\rceil \geq \sum_{i=0}^{k-1} \frac{k+1}{2^i} = (k+1) \sum_{i=0}^{k-1} \frac{1}{2^i} = \\ &= (k+1) \cdot \frac{2^k - 1}{2^{k-1}} > 2k + 1 \iff \\ \iff (k+1)(2^k - 1) &> (2k+1)2^{k-1} \iff 2^k > 2^{k-1} + k + 1 \end{aligned}$$

inequality which is true by induction for every $k \geq 4$ and thus from Griesmer's bound there is no linear code C .

- (ii) Let $q \geq 3$ and $n = qk + 1$.

- a) If $d \geq (q - 1)k + 3$, then $k + d \geq qk + 3 > qk + 2 = n + 1$ and thus from Singleton's bound there is no linear code C .

b) If $d = (q - 1)k + 2$, then:

$$\begin{aligned} \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil &= \sum_{i=0}^{k-1} \left\lceil \frac{(q-1)k+2}{q^i} \right\rceil \geq \sum_{i=0}^{k-1} \frac{(q-1)k+2}{q^i} = [(q-1)k+2] \sum_{i=0}^{k-1} \frac{1}{q^i} = \\ &= [(q-1)k+2] \cdot \frac{q^k-1}{q^{k-1}(q-1)} > qk+1 \iff \\ \iff [(q-1)k+2](q^k-1) &> (qk+1)q^{k-1}(q-1) \iff \\ \iff (q+1)q^{k-1} &> (q-1)k+2 \end{aligned}$$

inequality which is true by induction for every $k \geq 2$ and thus from Griesmer's bound there is no linear code C .

c) If $d = (q - 1)k + 1$, then:

$$\begin{aligned} \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil &= \sum_{i=0}^{k-1} \left\lceil \frac{(q-1)k+1}{q^i} \right\rceil \geq \sum_{i=0}^{k-1} \frac{(q-1)k+1}{q^i} = [(q-1)k+1] \sum_{i=0}^{k-1} \frac{1}{q^i} = \\ &= [(q-1)k+1] \cdot \frac{q^k-1}{q^{k-1}(q-1)} > qk+1 \iff \\ \iff [(q-1)k+1](q^k-1) &> (qk+1)q^{k-1}(q-1) \iff \\ \iff q^{k-1} &> (q-1)k+1 \end{aligned}$$

inequality which is true by induction for every $k \geq 3$ and thus from Griesmer's bound there is no linear code C .

In particular from theorem 3 we have the following two results:

Theorem 4. *There is no $[2k+1, k, k+1]$ -binary linear code for every $k \geq 4$.*

Theorem 5. *There is no $[3k+1, k, 2k+1]$ -ternary linear code for every $k \geq 3$.*

Moreover, we can extend theorem 3 to the following:

Theorem 6. *Let the parameters $[n, k, d]$ be such that $n, k, d \geq 1$ and $q \geq 2$. Then there is no linear code C over \mathbb{F}_q of parameters $[qk, k, (q-1)k+1]$ for every $k \geq 2$.*

Proof: We have the following inequality:

$$\begin{aligned} \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil &= \sum_{i=0}^{k-1} \left\lceil \frac{(q-1)k+1}{q^i} \right\rceil \geq \sum_{i=0}^{k-1} \frac{(q-1)k+1}{q^i} = [(q-1)k+1] \sum_{i=0}^{k-1} \frac{1}{q^i} = \\ &= [(q-1)k+1] \cdot \frac{q^k-1}{q^{k-1}(q-1)} > qk \iff \\ \iff [(q-1)k+1](q^k-1) &> kq^{k+1} - kq^k \iff q^k > (q-1)k+1 \end{aligned}$$

which is true for every $k \geq 2$.

In particular from theorem 6 we have the following two results:

Theorem 7. *There is no $[2k, k, k+1]$ -binary linear code for every $k \geq 2$.*

Theorem 8. *There is no $[3k, k, 2k+1]$ -ternary linear code for every $k \geq 2$.*

Remark 9. Theorems 7 and 8 are not true for $k = 1$ as one can see from below:

1) $C = \{00, 11\}$ is a $[2, 1, 2]$ -binary linear code.

2) $C = \{000, 111, 222\}$ is a $[3, 1, 3]$ -ternary linear code.

Moreover, if we consider the $[2k, k, k]$ -binary linear codes and $[3k, k, 2k]$ -ternary linear codes we have:

3) If $k = 1$, $C = \{00, 01\}$ is a $[2, 1, 1]$ -binary linear code.

3) If $k = 2$, $C = \{0000, 0011, 1100, 1111\}$ is a $[4, 2, 2]$ -binary linear code.

3) If $k = 3$,

$$C = \{000000, 111000, 011100, 001110, 100100, 110110, 010010, 101010\}$$

is a $[6, 3, 3]$ -binary linear code.

3) If $k = 1$, $C = \{000, 011, 022\}$ is a $[3, 1, 2]$ -ternary linear code.

Remark 10.

a. Regarding theorem 3, point i) there exist binary linear codes for $k \leq 3$ such as:

1) If $k = 1$ we have $C = \{000, 111\}$ a $[3, 1, 3]$ -binary linear code and $C = \{000, 011\}$ a $[3, 1, 2]$ -binary linear code.

2) If $k = 2$ we have $C = \{00000, 00111, 11100, 11011\}$ a $[5, 2, 3]$ -binary linear code.

3) If $k = 3$ we have

$$C = \{0000000, 0001111, 0110011, 0111100, 1100110, 1101001, 1010101, 1011010\}$$

a $[7, 3, 4]$ -binary linear code.

b. Regarding theorem 3, point ii) there exist ternary linear codes for $k \leq 2$ such as:

1) If $k = 1$ we have $C = \{0000, 1111, 2222\}$ a $[4, 1, 4]$ -ternary linear code and $C = \{0000, 0111, 0222\}$ a $[4, 1, 3]$ -ternary linear code.

2) If $k = 2$ we have

$$C = \{000000, 0011111, 0022222, 1110012, 2220021, 1102201, 2201102, 1121120, 2212210\}$$

a $[7, 2, 5]$ -ternary linear code.

c. If we consider the case of the binary linear codes of the type $[2k + 1, k, k]$, $k \geq 1$, one can have the following examples:

1) $C = \{000, 100\}$ is a $[3, 1, 1]$ -binary linear code.

2) $C = \{00000, 11000, 01100, 10100\}$ is a $[5, 2, 2]$ -binary linear code.

3)

$$C = \{0000000, 1110000, 0011100, 0000111, 1101100, \\ 1110111, 0011011, 1101011\}$$

is a $[7, 3, 3]$ -binary linear code.

4)

$$C = \{000000000, 111100000, 001111000, 000011110, 101010100, \\ 110011000, 001100110, 111111110, 110000110, 010110100, 100101100, \\ 101001010, 011001100, 100110010, 010101010, 011010010\}$$

is a $[9, 4, 4]$ -binary linear code

d. If we consider the case of the ternary linear codes of the type $[3k + 1, k, 2k]$, $k \geq 1$, one can have the following examples:

1) $C = \{0000, 1100, 2200\}$ is a $[4, 1, 2]$ -ternary linear code.

2)

$$C = \{0000000, 0001111, 0002222, 1110002, 2220001, 1111110, \\ 2222220, 1112221, 2221112\}$$

is a $[7, 2, 4]$ -ternary linear code.

References

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