Some Considerations About Component p-vertexes Complete Subgraph

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Abstract
In previous studies I approached component minimal complete subgraph for a given graph G. One can consider that the proper definition for component minimal subgraph is component 3-vertexes complete subgraph and so I consider that is natural to generalize it to component p-vertexes complete subgraph for any $p \geq 3$. To do this, in this paper I will introduce the general definition and some important results which will allow me in the future to build an algorithm for the determination of component p-vertexes complete subgraph for a given graph G
Keywords: *subgraphs, complete graphs*
ACM/AMS Classification: 05C30

1. General definitions

Firstly I wish to present some of the graph theory definitions.

Definition 1.1 *Let*

$$V = \{x_1, x_2, \ldots, x_n\}$$

be a finite and non-empty set and

$$E = \{\{x, y\} \,| x, y \in X, x \neq y\}.$$

The pair $G = (V, E)$ is a ***graph****, elements in V are named* ***vertices*** *and element in E are named* ***edge****.*

Definition 1.2 *Let $G = (V, E)$ be a graph. A sequence of vertices $y_1, y_2, \ldots, y_k$ is named* ***walk*** *in G if $\{y_i, y_{i+1}\} \in E$ for any $i = 1, 2, \ldots, k-1$. If $y_1 = y_k$ the walk is named* ***loop****. The walk (and the loop) is specified as*

$$L = [y_1, y_2, \ldots, y_k].$$

Definition 1.3 *Let $G = (V, E)$ be a graph. If $W \subset V$ and*

$$F = \{\{x, y\} \in E | x, y \in W\}$$

*then the graph $H = (W, F)$ is named **subgraph** of G.*

Definition 1.4 *Let $G = (V, E)$ be a graph. If for any $x, y \in V$ there exist a walk from x to y then G is named **connected**. If G is not connected, G is named **disconnected**.*

Definition 1.5 *Let $G = (V, E)$ be a graph. A subgraph $H = (W, F)$ of G which is connected and there does not exists a walk from x to y for any $x \in W$ and $y \in V - W$ is named **component** of G.*

Propozition 1.1 *Let $G = (V, E)$ be a graph. There exist a partition of V with the sets $V_1, V_2, \ldots, V_k$ ($V_1 \cup V_2 \cup \ldots \cup V_k = V$ and $V_i \cap V_j = \oslash$ for any $i, j = 1, 2, \ldots, k$, $i \neq j$) so that subgraphs $G_i = (V_i, F_i)$ are components in G.*

Definition 1.6 *Let $G = (V, E)$ be a graph with $|V| = p \leq 3$. If for any $x, y \in V$, $\{x, y\} \in E$ then G is named **complete** and we write it as K_p.*

Definition 1.7 *Let $G = (V, E)$ be a graph and $x \in V$. If we consider the set $C = \{y \in V | \{x, y\} \in E\}$ then the number $\omega(x) = |C|$ is named the **degree** of x.*

Definition 1.8 *Let $G = (V, E)$ be a connected graph. If in G does not exist loops then G is named **tree**.*

2. Component minimal complete graphs

This section is dedicated to a review for previous definitions and results dedicated to component minimal complete graphs which are introduced in Bârză 2012 [3].

Definition 2.1 *Let $G = (V, E)$ be a graph. If for any $x \in V$ there exist $y, z \in V$, $y \neq z$ so that $\{\{x, y\}, \{x, z\}, \{y, z\}\} \subset E$ we say that G is a **minimal complete graph**.*

Definition 2.2 *Let $G = (V, E)$ be a graph. If any component in G is a minimal complete graph then G is named **component minimal complete graph**.*

Let CMC_G designate the component minimal complete subgraph for the graph G, if such a subgraph exists.

Condition 2.1. *Let $G = (V, E)$ be a graph. We consider the set*

$$X = \{x \in V | \omega(x) = 0\}.$$

If $X = V$ then the determination of component minimal complete subgraph of G has no solution.

Proposition 2.1. *Let $G = (V, E)$ be a graph and*

$$X = \{x \in V | \omega(x) = 0\}$$

with $V - X \neq \oslash$. Let consider the subgraph $H = (V - X, F)$ of G. Then, G has a component minimal complete subgraph if and only if H has a component minimal complete subgraph.
In addition, we have

$$CMC_G = CMC_H.$$

Condition 2.2. *Let $G = (V, E)$ be a graph with components G_1, G_2, ..., G_k. If for any $i = 1, 2, \ldots, k$ G_i is a tree then the determination of component minimal complete subgraph of G has no solution.*

Proposition 2.2. *Let $G = (V, E)$ be a graph with components G_1, G_2, ..., G_k. We consider the set*

$$I = \{i | G_i \text{ is a tree}\},$$

$$X = \bigcup_{i \in I} V_i$$

and the subgraph $H = (V - X, F)$ of G. Then, G has a component minimal complete subgraph if and only if H has a component minimal complete subgraph.

In addition, we have

$$CMC_G = CMC_H.$$

Definition 2.3 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \oslash$. If $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, then the graph $G = (V, E)$ is* ***reunion graph*** *of G_1 and G_2 and we write*

$$G = G_1 \cup G_2.$$

We can generalize definition 2.3 to a finite number of graphs $G_i = (V_i, E_i)$ with $i = 1, 2, \ldots, n$ for which $V_i \cap V_j = \oslash$ for any $i \neq j$, $i, j = 1, 2, \ldots, n$. We now can consider the sets

$$V = \bigcup_{i=1}^{n} V_i = \bigcup_{1 \leq i \leq n} V_i$$

and

$$E = \bigcup_{i=1}^{n} E_i$$

and the graph $G = (V, E)$ is the reunion of finite family of graphs $G_1, G_2, \ldots, G_n$ and we write

$$G = \bigcup_{i=1}^{n} G_i.$$

Now let consider a graph G without isolated vertices and without tree components. We can give the following result:

Proposition 2.3. *Let $G = (V, E)$ be a graph without isolated vertices and without tree components. We consider that $G_1, G_2, \ldots, G_k$ are the components of G. If there exist i, $1 \leq i \leq k$ for which G_i has a component minimal complete subgraph, then G has a component minimal complete subgraph and if*

$$J = \{i | \textit{it exists } CMC_{G_i} \textit{ for } G_i\}$$

then

$$CMC_G = \bigcup_{i \in J} CMC_{G_i}.$$

Condition 2.3. *Let $G = (V, E)$ be a minimal a minimal complete graph. For any $x \in V$,*

$$\omega(x) \geq 2.$$

Proposition 2.4. *Let $G = (V, E)$ be a connected graph and $X = \{x \in V | \omega(x) = 1\}$. Let us consider the subgraph $H = (V - X, F)$ of G. Then, G gas a component minimal complete subgraph if and only if H has a component minimal complete subgraph and*

$$CMC_G = CMC_H.$$

Condition 2.4. *Let $G = (V, E)$ be a minimal complete graph. If $x \in V$ has $\omega(x) = 2$ and $y, z \in V$ are the vertices for which $\{\{x, y\}, \{x, z\}\} \subset E$, then $\{y, z\} \in E$.*

Proposition 2.5. *Let $G = (V, E)$ be a connected graph and*

$$X = \{x \in V | \omega(x) = 2,\ \{x, y\}, \{x, z\} \in E \text{ and } \{y, z\} \notin E\}.$$

Let consider the subgraph $H = (V - X, F)$ of G. Then, G has a component minimal complete subgraph if and only if H has a component minimal complete subgraph and

$$CMC_G = CMC_H.$$

Condition 2.5. *Let $G = (V, E)$ be a minimal complete graph. Then it does not exists $x, y \in V$ with $\{x, y\} \in E$ so that partial graph $H = (V, E - \{\{x, y\}\})$ of G is a disconnected graph.*

Proposition 2.6. *Let $G = (V, E)$ be a connected graph and*

$$T = \left\{u \in E | \bar{G} = (V, E - \{u\}) \text{ is disconnected}\right\}.$$

We consider partial graph $\bar{G} = (V, E - T)$ of G. Let

$$X = \left\{x \in V | \omega(x) = 0 \text{ in } \bar{G}\right\}$$

and the subgraph $H = (V - X, F)$ of $\bar{G}$. Then, G has a component minimal complete subgraph if and only if H has a component minimal complete subgraph and

$$CMC_G = CMC_H.$$

The problem of the determination of CMC_G is solved for sets approach in Bârză 2013 [4] and Bârză 2014 [5] and olso for algebraic approach in Bârză 2014 [6].

Because CMC_G if formed with complete subgraphs with 3 vertexes, in the future we will call them ***component* 3*-vertexes complete subgraph of*** G and we will use the notation $CK_3(G)$.

To identify better $CK_3(G)$ we will consider that, for a given graph $G = (V, E)$,

$$CK_3(G) = (V_3, E_3, \mathcal{C}K_3)$$

where $V_3 \subset V$ represents the vertexes of $CK_3(G)$, $E_3 \subset E$ is the set of edges for $CK_3(G)$ and $\mathcal{C}K_3$ represents the family of all sets of three vertexes $T = \{x, y, z\}$ so that the complete graph $(\{x, y, x\}, \{\{x, y\}, \{x, z\}, \{y, z\}\})$ is a subgraph of G.

3. **Component p-vertexes complete graphs**

In this section we will generalize the definition of component 3-vertexes complete graph and we will give some rezults on component p-vertexes complete graph, for $p \geq 3$.

Definition 3.1 *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V, E)$ be a graph. If for any $x \in V$ there exist $x_1, x_2, \ldots, x_{p-1} \in V$, with $x_i \neq x_j$ for any $1 \leq i \leq p-1$, $1 \leq j \leq p-1$, $i \neq j$, so that $\{\{y, z\} | y, z \in \{x, x_1, x_2, \ldots, x_{p-1}\}, \ , y \neq z\} \subset E$ we say that G is a* ***p-vertexes complete graph****. For a better identification of this graf we will give G as $G = (V, E, \mathcal{C}K_p)$, where $\mathcal{C}K_p$ is the family if all sets T so that the graph $H = (T, \{\{x, y\} | x \neq y, \ x, y \in T\})$ is a complete graph with p vertexes and is a subgraph in the graph (V, E).*

Note 3.1

One can see that Definition 3.1 is consistent with Definition 2.1 because in case $p = 3$ the condition $\{\{y, z\} | y, z \in \{x, x_1, x_2, \ldots, x_{p-1}\}, \ , y \neq z\} \subset E$ means $\{\{x, x_1\}, \{x, x_2\}, \{x_1.x_2\}\} \subset E$.

Definition 3.2 *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V, E)$ be a graph. If any component in G is a p-vertexesl complete graph then G is named* ***component p-vertexes complete graph****.*

Let $CK_p(G)$ designate the component p-vertexes complete subgraph for the graph $G = (V, E)$, if such a subgraph exists. We give $CK_p(G)$ as

$$CK_p(G) = (V_p, E_p, \mathcal{C}K_p),$$

where $V_p \subset V$ is the set of vertexes for component p-vertexes complete subgraph, $E_p \subset E$ represent the set of edges of component p-vertexes complete subgraph, and $\mathcal{C}K_p$ is the family al all sets T so that the graph

$$(T, \{\{x, y\} | x \neq y, \ x, y \in T\})$$

is a complete graph with p vertexes and it is a subgraph of G.

Note 3.2

One can see that this way to specify component p-vertexes complete subgraph is consistent with the form proposed for component 2-vertexes complete subgraph at the end of section 2.

We can consider that, if a graph G has not component p-vertexes complete subgraph, then $CK_p(G) = \emptyset = (\emptyset, \emptyset, \emptyset)$, so $V_p = \emptyset$, $E_p = \emptyset$, and $CK_p = \emptyset$.

Proposition 3.1. *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V, E)$ be a conected graph. If G is a tree, then $CK_p(G) = \emptyset$*

To work with component p-vertexes complete subgraph we extend the Definition 2.3 to

Definition 3.3 *Let $p \in \mathbb{N}$, $p \geq 3$, and $G^1 = (V_p^1, E_p^1, \mathcal{C}K_p^1)$ and $G^2 = (V_p^2, E_p^2, \mathcal{C}K_p^2)$ be two component p-vertexes complete graphs with $V^1 \cap V^2 = \oslash$. If $V_p = V_p^1 \cup V_p^2$, and $E_p = E_p^1 \cup E_2$, and $\mathcal{C}K_p = \mathcal{C}K_p^1 \cup \mathcal{C}K_p^2$, then the graph $G = (V_p, E_p, \mathcal{C}K_p)$ is **reunion component p-vertexes complete graph** of G^1 and G^2 and we write*

$$G = G^1 \cup G^2.$$

Like in Section 2, we can generalize definition 3.3 to a finite number of component p-vertexes complete graphs $G^i = (V_p^i, E_p^i, \mathcal{C}K_p^i)$ with $i = 1, 2, \ldots, n$ for which $V^i \cap V^j = \oslash$ for any $i \neq j$, $i, j = 1, 2, \ldots, n$. We now can consider the sets

$$V_p = \bigcup_{i=1}^{n} V_p^i = \bigcup_{1 \leq i \leq n} V_p^i,$$

and

$$E_p = \bigcup_{i=1}^{n} E_p^i,$$

and

$$\mathcal{C}K_p = \bigcup_{i=1}^{n} \mathcal{C}K_p^i,$$

The component p-vertexes complete graph $G = (V_p, E_p, \mathcal{C}K_p)$ is the ***reunion of finite family of component p-vertexes complete graphs*** $G^1, G^2, \ldots, G^n$ and we write

$$G = \bigcup_{i=1}^{n} G^i.$$

Proposition 3.2. *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V, E)$ be a graph with components G_1, G_2, ..., G_k (and so $G = \bigcup_{1 \leq i \leq k} G_i$. We consider the set*

$$I = \{i | G_i \text{ is not a tree}\},$$

and the subgraph

$$H = \bigcup_{i \in I} G_i$$

of G. Then, G has a component p-vertexes complete subgraph if and only if H has a p-vertexes complete subgraph.

In addition, we have

$$CK_p(G) = CK_p(H).$$

Note 3.3
One can see that, considering $p = 3$ and replacing CK_p with CMC, the Proposition 3.2 is consistent with Proposition 2.2.

Proposition 3.3. *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V_p, E_p, \mathcal{C}K_p)$ be a p-vertexes complete graph. For any $x \in V_p$,*

$$\omega(x) \geq p - 1.$$

The Proposition 3.3 is obvious true because, from definition, for any $x \in V_p$ there exists $x_1, x_2, \ldots, x_p - 1 in V_p$ which form a complete graph with x. So, for any $1 \leq i \leq p-1$, $\{x, x_i\} in E_p$, and the number of this edges in $p-1$. Because we can not exclude other possible edges formed with vertex x, it means that $\omega(x) \geq p-1$.

Proposition 3.4. *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V, E)$ be a connected graph and $X = \{x \in V | \omega(x) \leq p-2\}$. Let us consider the subgraph $H = (V - X, F)$ of G. Then, G has a component p-vertexes complete subgraph if and only if H has a component p-vertexes complete subgraph and*

$$CK_p(G) = CK_p(H).$$

Proposition 3.4 is true as a consequence of proposition 3.3.

Note 3.4
On can see that Proposition 3.4 include in the set X the isolated vertexes (which for the case of $p = 3$ is solved in Proposition 2.1. Also, Proposition 3.4 is consistent with Proposition 2.4, because in the case $p = 3$ we have $p-2 = 1$.

The results we presented till now in this section can allow us to think to the determination of component p-vertexes complete subgraph for a given graph G starting directly from G and firstly, using the above proposition to eliminate all the vertexes that can not participate in forming $CK_p(G)$. This way of solving the determination of component p-vertexes complete subgraph is not a proper one, because it does not take in consideration the possible connection between $CK_p(G)$ which is to be determined and already determined $CK_s(G)$ for $s = 3, \ldots, p-1$.

We consider here that the determination depends only by the connection between $CK_p(G)$ and $CK_{p-1}(G)$ and this is the fact which will be shown in the rest of this section.

Proposition 3.5. *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V, E)$ be a graph and $CK_{p-1}(G) = (V_{p-1}, E_{p-1}, \mathcal{C}K_{p-1})$ be the component $(p-1)$-vertexes complete subgrph of G, $CK_{p-1}(G) \neq \emptyset$. If we consider the set*

$$X = \{x \in V_{p-1} | \omega(x) = p-2\}$$

and $CK_p(G) = (V_p, E_p, (C)K_p)$, the component p-vertexes complete subgraph of G, then

$$V_p \subseteq V_{p-1} \setminus X$$

$$E_p \subseteq \{\{x,y\} \in E_{p-1} | x \notin X \text{ and } y \notin X\}$$

and for any $x \in X$ and any $T \in \mathcal{C}K_{p-1}$, $x \notin T$.

Proofs

Let $y \in V_p$. From Proposition 3.3. we have that $\omega(y) \geq p-1$ and so $y \notin X$. Because $y \in V_p$, it results that it exists $T \in \mathcal{C}K_p$ so that $y \in T$ and T identifies the graph $(T, \{\{s,t\} | s,t \in T\}$ which is a complete graph in which for any vertex $x \in T$ we have $\omega(x) = p-1$. It results that if $x \in T$ then $x \notin X$.

In the same time, because the graph $(T, \{\{s,t\} | s,t \in T\}$ is complete and $y \in T$, it results that $T \setminus \{y\}$ has $p-1$ elements and if we consider $z \in T$, $z \neq y$, the graph $(T \setminus \{z\}, \{\{s,t\} | s,t \in T \setminus \{z\}\}$ is a complete graph with $p-1$ vertexes and $y \in T \setminus \{z\}$, and so $y \in V_{p-1}$.

Because $y \notin X$ and $y \in V_{p-1}$ we can write $y \in V_{p-1} \setminus X$. Because we take $y \in V_p$ arbitrarly, it means that

$$V_p \subseteq V_{p-1} \setminus X.$$

The rest of the statements in Proposition 3.5. is obvious using

$$V_p \subseteq V_{p-1} \setminus X.$$

From the proofs of the relation $V_p \subseteq V_{p-1} \setminus X$ we can see that:

Proposition 3.6. *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V,E)$ be a graph and $CK_a(G) = (V_a, E_a, \mathcal{C}K_a)$ be a component a-vertexes complete subgraph of G, $a = p-1, p$. Then for any $T \in \mathcal{C}K_p$ we have*

$$\{T \setminus \{x\} | x \in T\} \subset \mathcal{C}K_{p-1}.$$

To present the next result we consider that, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs so that G_2 is subgraph of G_1, then G_2 is ***included*** in G_1 and we can write $G_2 \subseteq G_1$.

From Proposition 3.5. we can give too:

Proposition 3.7. *Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V,E)$ be a graph and $CK_a(G) = (V_a, E_a, \mathcal{C}K_a)$ be a component a-vertexes complete subgraph of G, $a = p-1, p$. Then*

$$(V_p, E_p) \subseteq (V_{p-1}, E_{p-1}).$$

Notation. Let $p \in \mathbb{N}$, $p \geq 3$, and $G = (V,E)$ be a graph and $CK_a(G) = (V_a, E_a, \mathcal{C}K_a)$ be a component a-vertexes complete subgraph of G, $a = p-1, p$. Then we can say that $CK_p \subseteq CK_{p-1}$.

We can use this notation, because from Proposition 3.5. we have $V_p \subseteq V_{p-1}$ and $E_p \subseteq E_{p-1}$.

Proposition 3.8. *Let $G = V, E)$ be a graph with n vertexes. Then*

$$\emptyset = CK_{n+1}(G) \subseteq CK_n(G) \subseteq \ldots \subseteq CK_4(G) \subseteq CK_3(G) \subseteq (V, E, E)$$

In addition, there exists p, $3 \le p \le n+1$, so that for any $q \ge p$, $CK_q(G) = \emptyset$

The affirmation in Proposition 3.8. is obvious using Proposition 3.7.

Proposition 3.8. says that $(CK_p)_{1 \le p \le n+1}$ represents a series of descending subgraphs of G which are component p-vertexes complete subgraphs of G and that, in the determination of this series we can stop at the first empty subgraphs.

The real connection between CK_{p-1} and CK_p is given by the next results, which is used previously in the algorithm for determination of component minimal complete subgraph for a given graph G.

Theorem 3.1. *Let $p \in \mathbb{N}$, $p \ge 3$, and $G = (V, E)$ be a graph and $CK_a(G) = (V_a, E_a, \mathcal{C}K_a)$ be a component a-vertexes complete subgraph of G, $a = p-1, p$.*

Let $T_1, T_2 \in \mathcal{C}K_{p-1}$ for which $T_1 \cap T_2$ has $p-2$ elements. Let $x, y \in V_{p-1}$ so that

$$T_1 = (T_1 \cap T_2) \cup \{x\}$$

and

$$T_2 = (T_1 \cap T_2) \cup \{y\}.$$

If $\{x, y\} \in E_{p-1}$, then $T_1 \cup T_2 \in \mathcal{C}K_p$ (and also $T_1 \cup T_2 \subseteq V_p$).

4. **Conclusions**

The subject of this paper is dedicated to the generalization of the notion of the component minimal complete graphs. This generalization is done for determine a subgraph in which any vertex is in at least a complete graph formed with p vertexes from given graph G.

I do not consider here this algorithm. In a future paper I wish to specify algorithms to solve this problem, in terms of sets operation.

The results are important because we can use them in the future to determine all maximal complete subgraphs for a given graph G, maximal complete subgraphs being known as *cliques*, and it represents an important part of the *perfect graphs* in graph theory.

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