# The Structure of $\lambda$-Measures on the Code Space

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**Abstract**
*We describe the concrete structure of the normalized $\lambda$-measures on the code space generated by a finite alphabet.*
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## 1. Introduction

While classical measure theory deals with additive (or countably additive measures), generalized measure theory studies measures which are not additive. These measures can describe in a more adequate manner concrete phenomena: super-additive measures describe cooperation, sub-additive measure describe lack of cooperation, while additive measures describe non interaction.

The most popular non-additive measures are the $\lambda$-additive measures (more particular, the $\lambda$-measures) introduced by the Japanese scholar Michio Sugeno in his doctoral thesis [5], which are intensely studied now. As pointed out in [1], these measures appear naturally also from functional point of view.

Normalized $\lambda$-measures (initially called *fuzzy measures* by M. Sugeno and called *Sugeno measures* today) have great impact in many domains. For instance, they are special cases of belief or plausibility measures (see [4] and [7]).

The problem of the concrete description of $\lambda$-measures on certain particular spaces is very important now. For instance, the structure of $\lambda$-measures with pre-assigned values on $\mathcal{P}(\mathbb{N})$ was completely solved in [2].

In the present paper, we shall describe the Sugeno measures on the code space generated by a finite alphabet (detailed definitions will be given in the next paragraph), with hints for not normalized $\lambda$-measures on the same code space.

## 2. **Preliminary Facts**

We shall denote by $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$ the set of (non null) natural numbers. The positive reals will be $\mathbb{R}_+ = [0, \infty)$. All sequences $(x_n)_n$ will be indexed with $\mathbb{N}$. When writing $(x_n)_n \subset A$, we mean $x_n \in A$ for any $n$.

First, some basic facts about the code space.

Let $2 \leq p \in \mathbb{N}$. We shall consider $p$ distinct elements $x_1, x_2, \ldots, x_p$ named letters. They form the ***alphabet*** $X = \{x_1, x_2, \ldots, x_p\}$. The most popular choice is $X = \{0, 1, \ldots, p-1\}$.

The ***free monoid generated by*** $X$ is $X^*$, consisting in all ***words*** of the form $x = u_1u_2\ldots u_n$ (where $u_i \in X$) with length $l(x) = n$. The empty word $e \in X^*$ with $l(e) = 0$ is also taken into consideration. We accept that $X \subset X^*$, i.e. any $x_i \in X$ may be viewed in $X^*$.

Let us write

$$X^\infty = X^{\mathbb{N}} = \{f : \mathbb{N} \to X\}$$

with elements $f \in X^\infty$ identified as follows:

$$f \equiv x \equiv u_1u_2\ldots u_n\ldots$$

where $u_n = f(n) \in X$ for any $n \in \mathbb{N}$. The elements of $X^\infty$ (sequences of elements in $X$) are called ***codes*** and $X^\infty$ is called the ***code space***.

The canonical topology of $X^\infty$ is introduced as follows. First, we consider the (metrizable and compact) topological space $(X, \tau)$, where $\tau$ is the discrete topology. Writing $(X_n, \tau_n) = (X, \tau)$ for any $n \in \mathbb{N}$, we can consider the topological product of all $(X_n, \tau_n)$ which is $(\prod_{n=1}^{\infty} X_n, \mathcal{U})$ where $\prod_{n=1}^{\infty} X_n = X^\infty$ and $\mathcal{U}$ is the product topology of the topologies $\tau_n$. Then $(X^\infty, \mathcal{U})$ is a metrizable and compact topological space.

We can describe a (countable) base for $\mathcal{U}$. Namely, for any $x \in X^*$, we define the set $xX^\infty$ as follows. If $x = e$, $eX^\infty = X^\infty$. If $x = u_1u_2\ldots u_n$, $xX^\infty$ consists of all sequences $v = v_1v_2\ldots v_m\ldots$, with $v_1 = u_1$, $v_2 = u_2$, $\ldots$, $v_n = u_n$. It can be shown that $\mathcal{P} = \{xX^\infty | x \in X^*\}$ is a countable base for $\mathcal{U}$. The sets $xX^\infty$ are both open and compact. The Borel sets of $(X^\infty, \mathcal{U})$ will be denoted by $\mathcal{B}$.

Let us recall that $\mathcal{P}$ is a generalized semi-ring (it is a semi-ring if and only if $p = 2$) which generates $\mathcal{B}$. Two classical measures or Sugeno measures on $\mathcal{B}$ which coincide on $\mathcal{P}$ must be identical (the values determine a measure on $\mathcal{B}$).

Now, some basic facts about ***$\lambda$-measures and Sugeno measures***.

Let $T$ be a non empty set, with the Boolean $\mathcal{P}(T) = \{A | A \subset T\}$ and an algebra of sets $\mathcal{A} \subset \mathcal{P}(T)$. A monotone measure is a function $\mu : \mathcal{A} \to \mathbb{R}_+$, having the following properties:

(i) $\mu(\emptyset) = 0$.

(ii) For any $A$, $B$ in $\mathcal{A}$ such that $A \subset B$, one has $\mu(A) \leq \mu(B)$ (i.e. $\mu$ is increasing).

(iii) $\mu(T) = a > 0$.

We shall work only with monotone measures.

Now, let $\lambda \in (-\frac{1}{a}, \infty)$. We shall say that $\mu$ ***satifies the $\lambda$-rule*** if, for any $A$, $B$ in $\mathcal{A}$ such that $A \cap B = \emptyset$, one has

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B).$$

(of course, 0-rule means additivity).

More particular, we say that $\mu$ ***is a $\lambda$-measure*** if, for any disjoint sequence $(E_n)_n \subset \mathcal{A}$ such that $E = \bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{A}$, one has

$$\mu(E) = \begin{cases} \sum\limits_{n=1}^{\infty} \mu(E_n), & \text{if } \lambda = 0 \\ \frac{1}{\lambda}(\prod\limits_{n=1}^{\infty}(1 + \lambda\mu(E_n)) - 1), & \text{if } \lambda \neq 0 \end{cases}.$$

(of course, to say that $\mu$ is a 0-measure means to say that $\mu$ is $\sigma$-additive).

Notice that any $\lambda$-measure satisfies the $\lambda$-rule (and not conversely).

In case $\mu(T) = a = 1$ and $\mu$ is a $\lambda$-measure, we say that $\mu$ is a ***$\lambda$-Sugeno measure***. In case there exists $\lambda \in (-1, \infty)$ such that $\mu$ is a $\lambda$-Sugeno measure, we say that $\mu$ is a ***Sugeno measure***.

For any $0 \neq \lambda \in (-1, \infty)$, define the strictly increasing bijection $h_\lambda : [0, 1] \to [0, 1]$, via

$$h_\lambda(x) = \frac{(1 + \lambda)^x - 1}{\lambda}.$$

(we call $h_\lambda$ ***transfer function***).

Let $\mathcal{M}$ be the set of all probabilities on $\mathcal{A}$, i.e.

$$\mathcal{M} = \{\mu : \mathcal{A} \to \mathbb{R}_+ | \mu(T) = 1 \text{ and } \mu \text{ is } \sigma\text{-additive}\}.$$

For any $\lambda \in (-1, \infty) \setminus \{0\}$, write

$$\mathcal{S}_\lambda = \{m : \mathcal{A} \to \mathbb{R}_+ | m \text{ is a } \lambda\text{-Sugeno measure}\}.$$

Write also

$$\mathcal{S} = \bigcup_{\lambda \in (-1,\infty)\setminus\{0\}} \mathcal{S}_\lambda.$$

A classical theorem of Z. Wang (see [6] and also [7]) states that there exists a bijection $V_\lambda : \mathcal{M} \to \mathcal{S}_\lambda$, acting via $V_\lambda(\mu) = m$, where $m(A) = h_\lambda(\mu(A))$ for any $A \in \mathcal{A}$.

## 3. Results

Here, we denote (see previous paragraph)

$$\mathcal{M} = \{\mu : \mathcal{B} \to \mathbb{R}_+ | \mu(X^\infty = 1 \text{ and } \mu \text{ is } \sigma\text{-additive}\}.$$

We take for granted the following *matricial* description of $\mathcal{M}$.

Write $U_p = \{1, 2, \ldots, p\}$. Call 0-***distribution*** (on $X^\infty$) a sequence $(D(n))_n$ where:

$$\begin{aligned}
&D(1) = (a(1), a(2), \ldots, a(p)) = (a(i))_{1\le i\le p}\\
&D(2) = (a(i,j))_{1\le i\le p,\ 1\le j\le p}\\
&\vdots\\
&D(n) = (a(i_1, i_2, \ldots, i_n))_{1\le i_k\le p} \text{ for } k = 1, 2, \ldots, n\\
&(\text{hence } D(n) \text{ has } p^n \text{ elements})\\
&\vdots
\end{aligned}$$

such that

a) For any $n \in \mathbb{N}$ and any $(i_1, i_2, \ldots, i_n) \in U_p^n$ one has $a(i_1, i_2, \ldots, i_n) \ge 0$;

b) $\sum_{i=1}^{n} a(i) = 1$;

c) For any$n \in \mathbb{N}$ and any $(i_1, i_2, \ldots, i_n) \in U_p^n$, one has

$$a(i_1, i_2, \ldots, i_n) = \sum_{i=1}^{p} a(i_1, i_2, \ldots, i_n, i).$$

The most popular 0-distribution is the ***Bernoulli distribution*** which is defined using only $D(1)$: for any $n \ge 2$, the distribution $D(n)$ is defined via

$$a(i_1, i_2, \ldots, i_n) = a(i_1)a(i_2)\ldots a(i_n).$$

If, in particular, one has $a(i) = \frac{1}{p}$ for any $i$, we have the ***uniform distribution***.

And now, the promised matricial description. Denote by $\mathcal{D}_0$ the set of all 0-distribution. There exists a bijection $T_0 : \mathcal{D}_0 \to \mathcal{M}$, given as follow:

$$T_0((D(n))_n) = \mu$$

where $\mu : \mathcal{B} \to \mathbb{R}_+$ acts, for any $n \in \mathbb{N}$ and any $(i_1, i_2, \ldots, i_n) \in U_p^n$, via

$$\mu(x_{i_1} x_{i_2} \ldots x_{i_n} X^\infty) = a(i_1, i_2, \ldots, i_n)$$

(here $D(n) = (a(i_1, i_2, \ldots, i_n))_{1\le i_k\le p)}$.

This representation can be *extended* up to the following results which furnish the concrete representation of the Sugeno measures on $\mathcal{B}$.

The first basic definition is

**Definition 1.**

*Let $\lambda \in (-1,\infty)\setminus\{0\}$. We call **$\lambda$-distribution** a sequence $(D_\lambda(n))_n$, where*

$$\begin{array}{l} D_\lambda(1) = (a_\lambda(1), a_\lambda(2), \ldots, a_\lambda(p)) = (a_\lambda(i))_{1\le i\le p} \\ D_\lambda(2) = (a_\lambda(i,j))_{1\le i\le p,\ 1\le j\le p} \\ \vdots \\ D_\lambda(n) = (a_\lambda(i_1,i_2,\ldots,i_n))_{1\le i_k\le p} \text{ for } k=1,2,\ldots,n \\ \vdots \end{array}$$

*with the following properties*:

a) *For any $n\in\mathbb{N}$ and any $(i_1,i_2,\ldots,i_n)\in U_p^n$, one has*

$$\begin{array}{l} \text{if } -1<\lambda<0, \text{ then } 0<a_\lambda(i_1,i_2,\ldots,i_n)\le 1 \\ \text{if } \lambda>0, \text{ then } a_\lambda(i_1,i_2,\ldots,i_n)\ge 1. \end{array}$$

b) $\prod\limits_{i=1}^{p} a_\lambda(i) = \lambda+1.$

c) *For any $n\in\mathbb{N}$ and any $(i_1,i_2,\ldots,i_n)\in U_p^n$, one has*

$$a_\lambda(i_1,i_2,\ldots,i_n) = \prod_{i=1}^{p} a_\lambda(i_1,i_2,\ldots,i_n,i).$$

**Notation:** For any $\lambda\in(-1,\infty)\setminus\{0\}$, we denote by $\mathcal{D}_\lambda$ the set of all $\lambda$-distributions. This notation is consistent with the previous notation $\mathcal{D}_0$ for the 0-distributions.

**Theorem 2.**

*Let $\lambda\in(-1,\infty)\setminus\{0\}$. We have a bijection $T_\lambda:\mathcal{D}_\lambda\to\mathcal{S}_\lambda$, described as follows:*

(i) *Let $(D_\lambda(n))_n\in\mathcal{D}_\lambda$, where $D_\lambda(n) = (a_\lambda(i_1,i_2,\ldots,i_n))_{1\le i_k\le p}$ as previously.*
*Then $T_\lambda((D_\lambda(n))_n) = m$, where*

$$m(x_{i_1}x_{i_2}\ldots x_{i_n}X^\infty) = \frac{a_\lambda(i_1,i_2,\ldots,i_n)-1}{\lambda}.$$

(ii) *The inverse $Z_\lambda = T_\lambda^{-1}:\mathcal{S}_\lambda\to\mathcal{D}_\lambda$ acts via $Z_\lambda(m) = (D_\lambda(n))_n$, where $D_\lambda(n) = (a_\lambda(i_1,i_2,\ldots,i_n))_{1\le i_k\le p}$ with*

$$a_\lambda(i_1,i_2,\ldots,i_n) = 1+\lambda m(x_{i_1}x_{i_2}\ldots x_{i_n}X^\infty).$$

This result can be *globalized* for all $\lambda$ as follows.
The second basic definition is
**Definition 3.**
*A **general distribution** is a sequence $(P(n))_n$, where*

$$\begin{array}{l} P(1) = (b(1), b(2), , b(p)) = (b(i))_{1 \le i \le p} \\ P(2) = (b(i,j))_{1 \le i \le p,\ 1 \le j \le p} \\ \vdots \\ P(n) = (b(i_1, i_2, \dots, i_n))_{1 \le i_k \le p} \text{ for } k = 1, 2, \dots, n \\ \vdots \end{array}$$

*with the following properties:*

(i) $\prod_{i=1}^{p} b(i) = b \in (0, \infty) \setminus \{1\}$.

(ii) *For any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$, one has*

$$\begin{array}{l} \text{if } 0 < b < 1, \text{ then } 0 < b(i_1, i_2, \dots, i_n) \le 1; \\ \text{if } b > 1, \text{ then } b(i_1, i_2, \dots, i_n) \ge 1 \end{array}$$

(iii) *For any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$, one has*

$$b(i_1, i_2, \dots, i_n) = \prod_{i=1}^{p} b(i_1, i_2, \dots, i_n, i).$$

**Notation:** We shall denote by $\mathcal{D}$ the set of all general distributions.
**Lemma 4.**
*One has the equality*

$$\mathcal{D} = \bigcup_{\lambda \in (-1, \infty) \setminus \{0\}} \mathcal{D}_\lambda$$

(*the union is disjoint*).

Using Lemma 4, we can define the function $T : \mathcal{D} \to \mathcal{S}$, via $T((P(n))_n) = T_\lambda((P(n))_n)$, if $(P(n))_n \in \mathcal{D}_\lambda$.

It is useful to explain the action of $T$, as follows:

a) Take $(P(n))_n \in \mathcal{D}$ arbitrarily.

b) If $P(1) = (b(i))_{1 \le i \le p}$, compute $b = \prod_{i=1}^{p} b(i) \in (0, \infty) \setminus \{1\}$.

c) Obtain $\lambda \in (-1, \infty) \setminus \{0\}$ from the equation $\lambda + 1 = b \Leftrightarrow \lambda = b - 1$. It follows that $(P(n))_n \in \mathcal{D}(\lambda)$.

d) Finally define

$$T((P(n))_n) = T_\lambda((P(n))_n).$$

The function $T$ is the globalization of all $T_\lambda$, as follows from

**Theorem 5.**

*The function $T$ is surjective and not injective. Namely, for any $x \in X^\infty$, the set $T^{-1}(\{\delta_x\})$ is infinite, where $\delta_x$ is the Dirac measure concentrated at $x$.*

## 4. Final Considerations

Up to now, we studied only Sugeno measures, which are $\lambda$-measures $\mu$ with $\mu(T) = 1$ (i.e. they are normalized). Detailed computations for the previous facts can be found in [3].

It is possible to give a similar matricial representation for $\lambda$-measures $\mu$ on the code space with $\mu(T) = A$, where $A > 0$. The formulae are more complicated, involving a transfer function of the form $h_\lambda : [0,1] \to [0,A]$, given via

$$h_\lambda(x) = \frac{(1+\lambda A)^x - 1}{\lambda}$$

for $\lambda \in (-\frac{1}{A}, \infty)$, $\lambda \neq 0$.

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