

The Structure of λ -Measures on the Code Space

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Abstract

We describe the concrete structure of the normalized λ -measures on the code space generated by a finite alphabet.

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1. Introduction

While classical measure theory deals with additive (or countably additive measures), generalized measure theory studies measures which are not additive. These measures can describe in a more adequate manner concrete phenomena: super-additive measures describe cooperation, sub-additive measure describe lack of cooperation, while additive measures describe non interaction.

The most popular non-additive measures are the λ -additive measures (more particular, the λ -measures) introduced by the Japanese scholar Michio Sugeno in his doctoral thesis [5], which are intensely studied now. As pointed out in [1], these measures appear naturally also from functional point of view.

Normalized λ -measures (initially called *fuzzy measures* by M. Sugeno and called *Sugeno measures* today) have great impact in many domains. For instance, they are special cases of belief or plausibility measures (see [4] and [7]).

The problem of the concrete description of λ -measures on certain particular spaces is very important now. For instance, the structure of λ -measures with pre-assigned values on $\mathcal{P}(\mathbb{N})$ was completely solved in [2].

In the present paper, we shall describe the Sugeno measures on the code space generated by a finite alphabet (detailed definitions will be given in the next paragraph), with hints for not normalized λ -measures on the same code space.

2. Preliminary Facts

We shall denote by $\mathbb{N} = \{1, 2, \dots, n, \dots\}$ the set of (non null) natural numbers. The positive reals will be $\mathbb{R}_+ = [0, \infty)$. All sequences $(x_n)_n$ will be indexed with \mathbb{N} . When writing $(x_n)_n \subset A$, we mean $x_n \in A$ for any n .

First, some basic facts about the code space.

Let $2 \leq p \in \mathbb{N}$. We shall consider p distinct elements x_1, x_2, \dots, x_p named letters. They form the **alphabet** $X = \{x_1, x_2, \dots, x_p\}$. The most popular choice is $X = \{0, 1, \dots, p-1\}$.

The **free monoid generated by** X is X^* , consisting in all **words** of the form $x = u_1 u_2 \dots u_n$ (where $u_i \in X$) with length $l(x) = n$. The empty word $e \in X^*$ with $l(e) = 0$ is also taken into consideration. We accept that $X \subset X^*$, i.e. any $x_i \in X$ may be viewed in X^* .

Let us write

$$X^\infty = X^{\mathbb{N}} = \{f : \mathbb{N} \rightarrow X\}$$

with elements $f \in X^\infty$ identified as follows:

$$f \equiv x \equiv u_1 u_2 \dots u_n \dots$$

where $u_n = f(n) \in X$ for any $n \in \mathbb{N}$. The elements of X^∞ (sequences of elements in X) are called **codes** and X^∞ is called the **code space**.

The canonical topology of X^∞ is introduced as follows. First, we consider the (metrizable and compact) topological space (X, τ) , where τ is the discrete topology. Writing $(X_n, \tau_n) = (X, \tau)$ for any $n \in \mathbb{N}$, we can consider the topological product of all (X_n, τ_n) which is $(\prod_{n=1}^{\infty} X_n, \mathcal{U})$ where $\prod_{n=1}^{\infty} X_n = X^\infty$ and \mathcal{U} is the product topology of the topologies τ_n . Then (X^∞, \mathcal{U}) is a metrizable and compact topological space.

We can describe a (countable) base for \mathcal{U} . Namely, for any $x \in X^*$, we define the set xX^∞ as follows. If $x = e$, $eX^\infty = X^\infty$. If $x = u_1 u_2 \dots u_n$, xX^∞ consists of all sequences $v = v_1 v_2 \dots v_m \dots$, with $v_1 = u_1, v_2 = u_2, \dots, v_n = u_n$. It can be shown that $\mathcal{P} = \{xX^\infty | x \in X^*\}$ is a countable base for \mathcal{U} . The sets xX^∞ are both open and compact. The Borel sets of (X^∞, \mathcal{U}) will be denoted by \mathcal{B} .

Let us recall that \mathcal{P} is a generalized semi-ring (it is a semi-ring if and only if $p = 2$) which generates \mathcal{B} . Two classical measures or Sugeno measures on \mathcal{B} which coincide on \mathcal{P} must be identical (the values determine a measure on \mathcal{B}).

Now, some basic facts about **λ -measures and Sugeno measures**.

Let T be a non empty set, with the Boolean $\mathcal{P}(T) = \{A | A \subset T\}$ and an algebra of sets $\mathcal{A} \subset \mathcal{P}(T)$. A monotone measure is a function $\mu : \mathcal{A} \rightarrow \mathbb{R}_+$, having the following properties:

- (i) $\mu(\emptyset) = 0$.
- (ii) For any A, B in \mathcal{A} such that $A \subset B$, one has $\mu(A) \leq \mu(B)$ (i.e. μ is increasing).

(iii) $\mu(T) = a > 0$.

We shall work only with monotone measures.

Now, let $\lambda \in (-\frac{1}{a}, \infty)$. We shall say that μ *satisfies the λ -rule* if, for any A, B in \mathcal{A} such that $A \cap B = \emptyset$, one has

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B).$$

(of course, 0-rule means additivity).

More particular, we say that μ *is a λ -measure* if, for any disjoint sequence $(E_n)_n \subset \mathcal{A}$ such that $E = \bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$, one has

$$\mu(E) = \begin{cases} \sum_{n=1}^{\infty} \mu(E_n), & \text{if } \lambda = 0 \\ \frac{1}{\lambda} \left(\prod_{n=1}^{\infty} (1 + \lambda\mu(E_n)) - 1 \right), & \text{if } \lambda \neq 0 \end{cases}.$$

(of course, to say that μ is a 0-measure means to say that μ is σ -additive).

Notice that any λ -measure satisfies the λ -rule (and not conversely).

In case $\mu(T) = a = 1$ and μ is a λ -measure, we say that μ is a λ -**Sugeno measure**. In case there exists $\lambda \in (-1, \infty)$ such that μ is a λ -Sugeno measure, we say that μ is a **Sugeno measure**.

For any $0 \neq \lambda \in (-1, \infty)$, define the strictly increasing bijection $h_\lambda : [0, 1] \rightarrow [0, 1]$, via

$$h_\lambda(x) = \frac{(1 + \lambda)^x - 1}{\lambda}.$$

(we call h_λ *transfer function*).

Let \mathcal{M} be the set of all probabilities on \mathcal{A} , i.e.

$$\mathcal{M} = \{\mu : \mathcal{A} \rightarrow \mathbb{R}_+ \mid \mu(T) = 1 \text{ and } \mu \text{ is } \sigma\text{-additive}\}.$$

For any $\lambda \in (-1, \infty) \setminus \{0\}$, write

$$\mathcal{S}_\lambda = \{m : \mathcal{A} \rightarrow \mathbb{R}_+ \mid m \text{ is a } \lambda\text{-Sugeno measure}\}.$$

Write also

$$\mathcal{S} = \bigcup_{\lambda \in (-1, \infty) \setminus \{0\}} \mathcal{S}_\lambda.$$

A classical theorem of Z. Wang (see [6] and also [7]) states that there exists a bijection $V_\lambda : \mathcal{M} \rightarrow \mathcal{S}_\lambda$, acting via $V_\lambda(\mu) = m$, where $m(A) = h_\lambda(\mu(A))$ for any $A \in \mathcal{A}$.

3. Results

Here, we denote (see previous paragraph)

$$\mathcal{M} = \{\mu : \mathcal{B} \rightarrow \mathbb{R}_+ \mid \mu(X^\infty) = 1 \text{ and } \mu \text{ is } \sigma\text{-additive}\}.$$

We take for granted the following *matricial* description of \mathcal{M} .

Write $U_p = \{1, 2, \dots, p\}$. Call **0-distribution** (on X^∞) a sequence $(D(n))_n$ where:

$$\begin{aligned} D(1) &= (a(1), a(2), \dots, a(p)) = (a(i))_{1 \leq i \leq p} \\ D(2) &= (a(i, j))_{1 \leq i \leq p, 1 \leq j \leq p} \\ &\vdots \\ D(n) &= (a(i_1, i_2, \dots, i_n))_{1 \leq i_k \leq p} \text{ for } k = 1, 2, \dots, n \\ &\text{(hence } D(n) \text{ has } p^n \text{ elements)} \\ &\vdots \end{aligned}$$

such that

- a) For any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$ one has $a(i_1, i_2, \dots, i_n) \geq 0$;
- b) $\sum_{i=1}^n a(i) = 1$;
- c) For any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$, one has

$$a(i_1, i_2, \dots, i_n) = \sum_{i=1}^p a(i_1, i_2, \dots, i_n, i).$$

The most popular 0-distribution is the ***Bernoulli distribution*** which is defined using only $D(1)$: for any $n \geq 2$, the distribution $D(n)$ is defined via

$$a(i_1, i_2, \dots, i_n) = a(i_1)a(i_2) \dots a(i_n).$$

If, in particular, one has $a(i) = \frac{1}{p}$ for any i , we have the ***uniform distribution***.

And now, the promised matricial description. Denote by \mathcal{D}_0 the set of all 0-distribution. There exists a bijection $T_0 : \mathcal{D}_0 \rightarrow \mathcal{M}$, given as follow:

$$T_0((D(n))_n) = \mu$$

where $\mu : \mathcal{B} \rightarrow \mathbb{R}_+$ acts, for any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$, via

$$\mu(x_{i_1} x_{i_2} \dots x_{i_n} X^\infty) = a(i_1, i_2, \dots, i_n)$$

(here $D(n) = (a(i_1, i_2, \dots, i_n))_{1 \leq i_k \leq p}$).

This representation can be *extended* up to the following results which furnish the concrete representation of the Sugeno measures on \mathcal{B} .

The first basic definition is

Definition 1.

Let $\lambda \in (-1, \infty) \setminus \{0\}$. We call λ -**distribution** a sequence $(D_\lambda(n))_n$, where

$$\begin{aligned} D_\lambda(1) &= (a_\lambda(1), a_\lambda(2), \dots, a_\lambda(p)) = (a_\lambda(i))_{1 \leq i \leq p} \\ D_\lambda(2) &= (a_\lambda(i, j))_{1 \leq i \leq p, 1 \leq j \leq p} \\ &\vdots \\ D_\lambda(n) &= (a_\lambda(i_1, i_2, \dots, i_n))_{1 \leq i_k \leq p \text{ for } k = 1, 2, \dots, n} \\ &\vdots \end{aligned}$$

with the following properties:

a) For any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$, one has

$$\begin{aligned} \text{if } -1 < \lambda < 0, \text{ then } 0 < a_\lambda(i_1, i_2, \dots, i_n) &\leq 1 \\ \text{if } \lambda > 0, \text{ then } a_\lambda(i_1, i_2, \dots, i_n) &\geq 1. \end{aligned}$$

b) $\prod_{i=1}^p a_\lambda(i) = \lambda + 1$.

c) For any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$, one has

$$a_\lambda(i_1, i_2, \dots, i_n) = \prod_{i=1}^p a_\lambda(i_1, i_2, \dots, i_n, i).$$

Notation: For any $\lambda \in (-1, \infty) \setminus \{0\}$, we denote by \mathcal{D}_λ the set of all λ -distributions. This notation is consistent with the previous notation \mathcal{D}_0 for the 0-distributions.

Theorem 2.

Let $\lambda \in (-1, \infty) \setminus \{0\}$. We have a bijection $T_\lambda : \mathcal{D}_\lambda \rightarrow \mathcal{S}_\lambda$, described as follows:

- (i) Let $(D_\lambda(n))_n \in \mathcal{D}_\lambda$, where $D_\lambda(n) = (a_\lambda(i_1, i_2, \dots, i_n))_{1 \leq i_k \leq p}$ as previously. Then $T_\lambda((D_\lambda(n))_n) = m$, where

$$m(x_{i_1} x_{i_2} \dots x_{i_n} X^\infty) = \frac{a_\lambda(i_1, i_2, \dots, i_n) - 1}{\lambda}.$$

- (ii) The inverse $Z_\lambda = T_\lambda^{-1} : \mathcal{S}_\lambda \rightarrow \mathcal{D}_\lambda$ acts via $Z_\lambda(m) = (D_\lambda(n))_n$, where $D_\lambda(n) = (a_\lambda(i_1, i_2, \dots, i_n))_{1 \leq i_k \leq p}$ with

$$a_\lambda(i_1, i_2, \dots, i_n) = 1 + \lambda m(x_{i_1} x_{i_2} \dots x_{i_n} X^\infty).$$

This result can be *globalized* for all λ as follows.
The second basic definition is

Definition 3.

A **general distribution** is a sequence $(P(n))_n$, where

$$\begin{aligned} P(1) &= (b(1), b(2), \dots, b(p)) = (b(i))_{1 \leq i \leq p} \\ P(2) &= (b(i, j))_{1 \leq i \leq p, 1 \leq j \leq p} \\ &\vdots \\ P(n) &= (b(i_1, i_2, \dots, i_n))_{1 \leq i_k \leq p \text{ for } k = 1, 2, \dots, n} \\ &\vdots \end{aligned}$$

with the following properties:

(i) $\prod_{i=1}^p b(i) = b \in (0, \infty) \setminus \{1\}$.

(ii) For any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$, one has

$$\begin{aligned} \text{if } 0 < b < 1, \text{ then } 0 < b(i_1, i_2, \dots, i_n) &\leq 1; \\ \text{if } b > 1, \text{ then } b(i_1, i_2, \dots, i_n) &\geq 1 \end{aligned}$$

(iii) For any $n \in \mathbb{N}$ and any $(i_1, i_2, \dots, i_n) \in U_p^n$, one has

$$b(i_1, i_2, \dots, i_n) = \prod_{i=1}^p b(i_1, i_2, \dots, i_n, i).$$

Notation: We shall denote by \mathcal{D} the set of all general distributions.

Lemma 4.

One has the equality

$$\mathcal{D} = \bigcup_{\lambda \in (-1, \infty) \setminus \{0\}} \mathcal{D}_\lambda$$

(the union is disjoint).

Using Lemma 4, we can define the function $T : \mathcal{D} \rightarrow \mathcal{S}$, via $T((P(n))_n) = T_\lambda((P(n))_n)$, if $(P(n))_n \in \mathcal{D}_\lambda$.

It is useful to explain the action of T , as follows:

a) Take $(P(n))_n \in \mathcal{D}$ arbitrarily.

b) If $P(1) = (b(i))_{1 \leq i \leq p}$, compute $b = \prod_{i=1}^p b(i) \in (0, \infty) \setminus \{1\}$.

c) Obtain $\lambda \in (-1, \infty) \setminus \{0\}$ from the equation $\lambda + 1 = b \Leftrightarrow \lambda = b - 1$. It follows that $(P(n))_n \in \mathcal{D}(\lambda)$.

d) Finally define

$$T((P(n))_n) = T_\lambda((P(n))_n).$$

The function T is the globalization of all T_λ , as follows from

Theorem 5.

The function T is surjective and not injective. Namely, for any $x \in X^\infty$, the set $T^{-1}(\{\delta_x\})$ is infinite, where δ_x is the Dirac measure concentrated at x .

4. Final Considerations

Up to now, we studied only Sugeno measures, which are λ -measures μ with $\mu(T) = 1$ (i.e. they are normalized). Detailed computations for the previous facts can be found in [3].

It is possible to give a similar matricial representation for λ -measures μ on the code space with $\mu(T) = A$, where $A > 0$. The formulae are more complicated, involving a transfer function of the form $h_\lambda : [0, 1] \rightarrow [0, A]$, given via

$$h_\lambda(x) = \frac{(1 + \lambda A)^x - 1}{\lambda}$$

for $\lambda \in (-\frac{1}{A}, \infty)$, $\lambda \neq 0$.

References

1. I. Chişescu, *Why λ -additive (fuzzy) measures?*, Kybernetika, 51 no. 2 (2015), 246 – 254.
2. I. Chişescu, D. Stănică, A. Plăviţu, *Existence and uniqueness of countable λ -measures with preassigned values*, Fuzzy Sets and Systems, 244 (2014), 1 – 19.
3. I. Chişescu, A. Plăviţu, *Sugeno Measures on the Code Space*, Romanian Journal of Science and Technology, 17 no. 3 (2014), 288 – 300.
4. G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press, 1976.
5. M. Sugeno, *Theory of Fuzzy Integrals and Its Applications*, Ph. D. dissertation. Tokyo Institute of Technology, 1974.
6. Z. Wang, *Une classe de mesures floues – les quasi-mesures*, BUSEFAL, 6 (1981), 23 – 37.
7. Z. Wang, G. Klir, *Generalized Measure Theory*, Springer (IFSR International Series on Systems Science and Engineering 25), 2009.

