A Characterization of q–Gaussian Random Variables in Terms of a Wick Product Identity

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Abstract
If f and g are polynomials and X is a Gaussian random variable, then the Wick product of $f(X)$ and $g(X)$ can be written as an alternating sum of products of powers of the annihilation operator applied to $f(X)$ and $g(X)$. In each term the same power of the annihilation operator is applied to both $f(X)$ and $g(X)$. A natural question arises from this formula, namely, what are the random variables (or equivalently probability distributions), having finite moments of all orders, for which such a formula holds. We show that the answer to this question is: the q–Gaussians.
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1. **Introduction**

If X is a standard Gaussian random variable, and f and g are two polynomial functions, then the Wick product, $f(X) \diamond g(X)$, of $f(X)$ and $g(X)$, can be written as:

$$f(X) \diamond g(X) \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f^{(n)}(X) g^{(n)}(X), \tag{1}$$

where for each non–negative integer n and any polynomial function h, $h^{(n)}$ denotes the n–th derivative of h (see [7], [8], [14], and [15]). In the above formula, only finitely many terms of the series from the right are non–zero, since f and g are polynomial functions.

It is not hard to see that, for a standard Gaussian random variable X, the differentiation operator restricted to the space F of all random variables of the form $f(X)$, where f is a polynomial of one variable with complex coefficients, is equal to the annihilation operator a^-. Thus, formula (1) can be rewritten as:

$$f(X) \diamond g(X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(a^-\right)^n f(X) \left(a^-\right)^n g(X), \tag{2}$$

for all f and g polynomial functions. Because X is a standard Gaussian random variable, X is symmetric about the origin.

Since the annihilation operator can be defined for any random variable X having finite moments of all orders, it is natural to ask the following question:

What are the symmetric random variables X, having finite moments of all orders, for which there exists a sequence $\{c_n\}_{n\geq 0}$ of real numbers, such that, for any f and g polynomial functions, we have:

$$f(X) \diamond g(X) = \sum_{n=0}^{\infty} c_n \left(a^-\right)^n f(X) \left(a^-\right)^n g(X). \tag{3}$$

The purpose of this paper is to answer this question, finding also for each random variable, the corresponding sequence $\{c_n\}_{n\geq 0}$.

The paper is structured as follows. In section 2, we present a minimal background about the quantum operators: annihilation, preservation, and creation operators. In section 3 we review the definition of the Wick product generated by a random variable X having finite moments of all orders. In section 4, we review the q–Gaussian random variables. Finally in section 5 present the main result of the paper, which will be the answer to the above question.

2. **Background**

Let $(\Omega,\ \mathcal{F},\ P)$ be a probability space, and $X : \Omega \to \mathbb{R}$ a measurable function, i.e., X is a random variable. We assume that X has finite moments of all orders, i.e., for all $p > 0$, we have:

$$\begin{aligned} E\left[|f(X)|^p\right] &:= \int_\Omega |f(\omega)|^p dP(\omega) \\ &< \infty, \end{aligned} \tag{4}$$

where E denotes the expectation. Since X has finite moments of all orders, all the monomials $1,\ X,\ X^2,\ \ldots$ belong to the space $L^2(\Omega,\ \mathcal{F},\ P)$. Because this space is an inner product space, we can apply the Gram–Schmidt orthogonalization process to the sequence $1,\ X,\ X^2,\ \ldots$, obtaining a sequence of orthogonal polynomial random variables $f_0(X),\ f_1(X),\ f_2(X),\ \ldots$, such that, for all $n \geq 0$, we have:

$$\mathbb{C}1 + \mathbb{C}X + \cdots + \mathbb{C}X^n = \mathbb{C}f_0(X) + \mathbb{C}f_1(X) + \cdots + \mathbb{C}f_n(X). \tag{5}$$

Moreover, for any $n \geq 0$, if $f_n \neq 0$, then the degree of f_n is n, and we can choose f_n to have the leading coefficient 1.

If X takes on only finitely many values, k, with a positive probability, that means, if the probability distribution of X is a convex combination of k Dirac delta measures, then using Lagrange interpolation polynomial, it easy to see that for any function $f : \mathbb{R} \to \mathbb{C}$, there exists a polynomial g, of degree at most $k-1$, such that:

$$f(X) \quad = \quad g(X), \tag{6}$$

almost surely. Thus the space F of all random variables of the form $f(X)$, where f is a polynomial of one variable with complex coefficients, is equal to the space F_{k-1} of all random variables of the form $g(X)$, where g is a polynomial of degree at most $k-1$. Therefore, in this case we have:

$$f_k = f_{k+1} = f_{k+2} = \cdots = 0. \tag{7}$$

If the support of the probability distribution of X is an infinite set, then for all $n \geq 0$, $f_n \neq 0$, and thus f_n is a polynomial of degree n with leading coefficient 1.

For all non–negative integers n, we define the spaces:

$$F_n \quad := \quad \mathbb{C}f_0 + \mathbb{C}f_1 + \cdots + \mathbb{C}f_n, \tag{8}$$

and

$$G_n \quad := \quad \mathbb{C}f_n. \tag{9}$$

We also define:

$$F_{-1} = G_{-1} \quad := \quad \{0\}, \tag{10}$$

where $\{0\}$ denotes the null space.

We observe that F_n is the space of all random variables of the form $f(X)$, where f is a polynomial with complex coefficients, in one variable, of degree at most n, and

$$G_n \quad := \quad F_n \ominus F_{n-1}, \tag{11}$$

that means, G_n is the orthogonal complement of F_{n-1} into F_n, for all $n \geq 0$. For each $n \geq 0$, we cal, G_n the *homogenous chaos space of order* n, generated by X. We call every polynomial random variable $f(X)$ from $\cup_{n \geq 0} G_n$ a *homogenous polynomial*.

We are going to change the way that we view X, regarding it not as a random variable, but as a multiplication operator defined on the space F of all polynomial random variables with values in F, that maps:

$$f(X) \quad \mapsto \quad Xf(X), \tag{12}$$

for any polynomial function f. We denote this multiplication operator by the same letter X, as the random variable X. Due to the fact that the operator of multiplication by X is symmetric, it is not hard to see that we have the following lemma:

Lemma 2.1 *For all $n \geq 0$, if $\{f_n(X)\}_{n\geq 0}$ denote the orthogonal polynomial random variables generated by X, then:*

$$Xf_n(X) \quad \perp \quad f_k(X), \tag{13}$$

for all $k \neq n-1$, n, and $n+1$, where "$\perp$" means "orthogonal to".

For all $n \geq 0$, since $xf_n(x)$ is a polynomial of degree $n+1$, there exist real constants c_0, c_1, ..., c_{n+1}, such that:

$$Xf_n(X) \quad = \quad c_0 f_0(X) + c_1 f_1(X) + \cdots + c_{n+1} f_{n+1}(X), \tag{14}$$

but due to Lemma 2.1, we have:

$$c_0 = c_1 = \cdots = c_{n-2} \quad = \quad 0. \tag{15}$$

Thus we have:

$$Xf_n(X) \quad = \quad c_{n+1} f_{n+1}(X) + c_n f_n(X) + c_{n-1} f_{n-1}(X). \tag{16}$$

Since both $f_n(X)$ and $f_{n+1}(X)$ have the leading coefficient equal to 1, and $f_n(X)$ and $f_{n-1}(X)$ do not contain the term X^{n+1}, identifying the coefficients of X^{n+1} in both sides of formula (16), we conclude that:

$$c_{n+1} \quad = \quad 1, \tag{17}$$

for all $n \geq 0$. For all $n \geq 0$, we define the following numbers:

$$\alpha_n \quad := \quad c_n \tag{18}$$

and

$$\omega_n \quad := \quad c_{n-1}, \tag{19}$$

with the specification that for $n = 0$, since $f_{-1} := 0$, ω_0 can be any real number. We choose $\omega_0 := 0$.

The numbers $\{\alpha_n\}_{n\geq 0}$ and $\{\omega_n\}_{n\geq 1}$ are called the *Szegö–Jacobi parameters* of X. Relation (16) becomes now the *fundamental recursive relation of orthogonal polynomials:*

$$Xf_n(X) \quad = \quad f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}(X), \tag{20}$$

for all $n \geq 0$.
For each non–negative integer n, we define the following linear operators:

$$D_n^- : G_n \to G_{n-1}, \tag{21}$$

$$D_n^-(cf_n) \quad := \quad c\omega_n f_{n-1}. \tag{22}$$

Since D_n^- maps G_n into G_{n-1}, that means it decreases the degree of a homogenous polynomial by 1 unit, we call D_n^- an *annihilation operator.*

$$D_n^0 : G_n \to G_n, \tag{23}$$

$$D_n^0(cf_n) \;:=\; c\alpha_n f_n. \tag{24}$$

Since D_n^0 maps G_n into G_n, that means it preserves the degree of a homogenous polynomial, we call D_n^0 a *preservation operator.*

$$D_n^+ : G_n \to G_{n+1}, \tag{25}$$

$$D_n^+(cf_n) \;:=\; cf_{n+1}. \tag{26}$$

Since D_n^+ maps G_n into G_{n+1}, that means it increases the degree of a homogenous polynomial by 1 unit, we call D_n^+ a *creation operator.*

The fundamental recursive relation (20) can be written now as:

$$X|G_n \;=\; D_n^+ + D_n^0 + D_n^-, \tag{27}$$

where $X|G_n$ denotes the restriction of the multiplication operator by X to G_n, for all $n \geq 0$.

So far the creation, preservation, and annihilation operators have been defined only on each individual homogenous chaos space G_n, $n \geq 0$, but we can extend their definition by linearity, to the space of all polynomial random variables F, where

$$F \;:=\; \cup_{n\geq 0} F_n, \tag{28}$$

in the following way: for every polynomial random variable:

$$f(X) \;=\; c_0 f_0(X) + c_1 f_1(X) + c_2 f_2(X) + \cdots, \tag{29}$$

where only finitely many coefficients $\{c_n\}_{n\geq 0}$ are not zero, we define:

$$\begin{aligned} a^- f(X) &:= D_0^-(c_0 f_0) + D_1^-(c_1 f_1) + D_2^-(c_2 f_2) + \cdots & (30)\\ &= c_0\omega_0 f_{-1}(X) + c_1\omega_1 f_0(X) + c_2\omega_2 f_1(X) + \cdots, & (31)\end{aligned}$$

and call a^- the *annihilation operator*,

$$\begin{aligned} a^0 f(X) &:= D_0^0(c_0 f_0) + D_1^0(c_1 f_1) + D_2^0(c_2 f_2) + \cdots & (32)\\ &= c_0\alpha_0 f_0(X) + c_1\alpha_1 f_1(X) + c_2\alpha_2 f_2(X) + \cdots, & (33)\end{aligned}$$

and call a^0 the *preservation operator*, and

$$\begin{aligned} a^+ f(X) &:= D_0^+(c_0 f_0) + D_1^+(c_1 f_1) + D_2^+(c_2 f_2) + \cdots & (34)\\ &= c_0 f_1(X) + c_1 f_2(X) + c_2 f_3(X) + \cdots, & (35)\end{aligned}$$

and call a^+ the *creation operator* generated by X. We call a^-, a^0, and a^+ the *quantum operators* generated by the random variable X.

The fundamental recursive relation (20) becomes now:

Theorem 2.2 *The multiplication operator by X can be written as:*

$$X = a^- + a^0 + a^+. \tag{36}$$

In this equality the domain of X, a^-, a^0, and a^+ is understood to be the space F of all random variables of the form $f(X)$, where f is a polynomial of one variable with complex coefficients.

It is known that if $\{f_n(X)\}_{n\geq 0}$ represent the monic (that means the leading coefficient is equal to 1) orthogonal polynomial random variables generated by X, then for all $n \geq 1$, we have:

$$\| f_n(X) \|_2^2 = \omega_1\omega_2\cdots\omega_n. \tag{37}$$

It follows from this relation that:

- If the probability distribution of X has an infinite support, then for all $n \geq 1$, we have:

$$\omega_n > 0. \tag{38}$$

- If the probability distribution of X has a finite support of cardinality k, then for all $n \leq k-1$, we have:

$$f_n(X) \neq 0, \tag{39}$$

  while for all $n \geq k$,

$$f_n(X) = 0. \tag{40}$$

  Thus, for all $1 \leq n \leq k-1$, we have:

$$\omega_n > 0, \tag{41}$$

  while

$$\omega_k = 0. \tag{42}$$

In this case, the values of α_n, for $n \geq k$, and ω_n, for $n \geq k+1$, are irrelevant.

The reciprocal is also true, since we have the following theorem:

Theorem 2.3 (**Favard Theorem**)

- *If $\{\alpha_n\}_{n\geq 0}$ is a sequence of real numbers and $\{\omega_n\}_{n\geq 1}$ a sequence of positive real numbers, then there exists a random variable X, having finite moments of all orders, whose Szegö–Jacobi parameters are $\{\alpha_n\}_{n\geq 0}$ and $\{\omega_n\}_{n\geq 1}$.*

- *If $\{\alpha_n\}_{0\leq n\leq k-1}$ is a finite sequence of real numbers and $\{\omega_n\}_{1\leq n\leq k-1}$ is a finite sequence of positive real numbers, then there exists a random variable X, whose probability distribution has a finite support, of cardinality k, whose relevant Szegö–Jacobi parameters are $\{\alpha_n\}_{0\leq n\leq k-1}$ and $\{\omega_n\}_{1\leq n\leq k-1}$.*

The following theorem was proven in [1]:

Theorem 2.4 *If X is a random variable, having finite moments of all orders, then X is polynomially symmetric, i.e.,*

$$E\left[X^{2n-1}\right] = 0, \tag{43}$$

for all $n \in \mathbb{N}$, if and only if

$$a^0 = 0. \tag{44}$$

That means, for a polynomially symmetric random variable, we have:

$$X = a^- + a^+. \tag{45}$$

The notion of "polynomial symmetry" is a weaker form of symmetry. If X is *symmetric*, i.e., for all Borel subset B of $\mathbb{R}$, we have:

$$P(X \in B) = P(X \in -B), \tag{46}$$

where:

$$-B := \{-x \mid x \in B\}, \tag{47}$$

then X is also polynomial symmetric. In what follows we consider only polynomially symmetric random variables.

3. Wick product

If X is a random variable, having finite moments of all orders, and $\{f_n\}_{n\geq 0}$ is the sequence of orthogonal polynomials with leading coefficient 1, generated by X, then we define for all m and n non–negative integers:

$$f_m \diamond f_n := f_{m+n}. \tag{48}$$

We call $f_m \diamond f_n$, the *Wick product* of f_m and f_n. It must be noted that, if the probability distribution of X has a finite support of cardinality k, then for $m + n \geq k$, we have $f_{m+n} = 0$, and so,

$$f_m \diamond f_n = 0. \tag{49}$$

We extend the definition of the Wick product, from $F \times F$ to F, in the following linear way: if

$$f(X) = \sum_{n=0}^{\infty} c_n f_n(X) \tag{50}$$

and

$$g(X) \quad = \quad \sum_{n=0}^{\infty} d_n g_n(X), \tag{51}$$

where c_n and d_n are complex numbers, and only finitely many of them are non–zero, then we define:

$$f(X) \diamond g(X) \quad = \quad \sum_{k=0}^{\infty} \left[\sum_{m+n=k} c_m d_n \right] f_k(X). \tag{52}$$

The Wick product can be defined on larger spaces than $F \times F$. These spaces are defined in terms of the *second quantization operator* $\Gamma(cI)$, of c times the identity operator I, where c is a real constant.

The interested reader may consult the following papers:

- Sufficient conditions that guarantee that the Wick product of two L^2 random variables is an L^2 random variables for general probability measures can be found in [10].

- Sharp conditions that guarantee that the Gaussian Wick product of two L^2 random variables is an L^2 random variable can be found in [**?**]. Sharp conditions that guarantee that the Gaussian Wick product of two L^1 (L^∞) random variables is an $L^1(L^\infty)$ random variable can be found in [**?**]. Finally, sharp conditions that guarantee that the Gaussian Wick product of an L^p random variable and an L^q random variable is an L^r random variable can be found in [4].

- Sharp conditions that guarantee that the Wick product, generated by the square of a Gaussian random variable, of an L^p random variable and an L^q random variable is an L^r random variable can be found in [12].

- Almost sharp conditions that guarantee that the Poissonian Wick product of an L^p random variable and an L^q random variable is an L^r random variable can be found in [13].

In this paper, to keep things simple, we consider only polynomial random variables, that means random variables that can be written as a linear combination of finitely many orthogonal polynomials, and so there is no problem about the convergence of the series (52), which is a finite sum. Also, we are considering only one random variable (the one–dimensional case). The Wick product can be defined for more than one random variable, and in particular, the interested reader can find more about the q–Wick product, in the multi–dimensional case, in [3] and [6].

4. q–Gaussian random variables

If, in particular, one has $a(i) = \frac{1}{p}$ for any i, we have the ***uniform distribution***.

In this section we briefly review the q–Gaussian random variables.

Definition 4.1

A random variable X, having finite moments of all orders, is called a q–Gaussian random variable, where q is a fixed number in the interval $[-1, \infty)$, if it is polynomially symmetric and the q–commutator of its annihilation and creation operators, $[a^-, a^+]_q$, satisfies the following relation:

$$[a^-, a^+]_q = cI, \tag{53}$$

where:

$$[a^-, a^+]_q := a^-a^+ - qa^+a^-, \tag{54}$$

c is a real constant, and I denotes the identity operator of the space F of all random variables of the form $f(X)$, with f a polynomial of one variable with complex coefficients.

Since the q–Gaussian are symmetric random variables, all of their α–Szegö–Jacobi parameters are equal to zero, i.e., for all $n \geq 0$,

$$\alpha_n := 0. \tag{55}$$

Moreover, if $\{f_n(X)\}_{n\geq 0}$ denote the monic orthogonal polynomials generated by the q–Gaussian random variable X, then for all $n \geq 0$, we have:

$$\begin{aligned}
[a^-, a^+]_q f_n(X) &= a^-a^+f_n(X) - qa^+a^-f_n(X) \\
&= a^-f_{n+1}(X) - qa^+\left(\omega_n f_{n-1}(X)\right) \\
&= \omega_{n+1}f_n(X) - q\omega_n f_n(X) \\
&= \left(\omega_{n+1} - q\omega_n\right) f_n(X).
\end{aligned} \tag{56}$$

Since, for all $n \geq 0$, we have:

$$[a^-, a^+]_q f_n(X) = cf_n(X), \tag{57}$$

we conclude that:

$$\omega_{n+1} - q\omega_n = c, \tag{58}$$

for all $n < k$, where k denotes the cardinality of the support of the probability distribution of X. Iterating formula (58), since we can take $\omega_0 := 0$, we obtain for all $n < k$:

$$\begin{aligned}
\omega_n &= \sum_{i=1}^{n} q^{n-i}\left(\omega_i - q\omega_{i-1}\right) \\
&= \sum_{i=1}^{n} q^{n-i}c \\
&= c\left(1 + q + q^2 + \cdots + q^{n-1}\right).
\end{aligned} \tag{59}$$

Therefore, for all $0 \leq n < k$, we have:

$$\omega_n \quad := \quad \begin{cases} c\,(1-q^n)\,/\,(1-q) & \text{if} \quad q \neq 1 \\ \\ cn & \text{if} \quad q = 1. \end{cases} \tag{60}$$

From this relation, we can see that for $c = 0$, we have $\omega_n = 0$, for all $n \geq 0$, and so, in this case $X = 0$ (the identically zero random variable), which is not an interesting random variable.

For $c \neq 0$, since $\omega_1 = c$, we must have $c > 0$. In this case, it is not possible to have $q < -1$, since otherwise $\omega_2 < 0$, which contradicts Favard's theorem. Thus we must have $q \geq -1$.

For $q = -1$, we have $\omega_2 = 0$, and so the cardinality of the support of the probability distribution of X is $k = 2$. Thus X, takes on only two values with positive probability, and since X is symmetric, X must be a Bernoulli random variable. Since:

$$\begin{aligned} E\left[X^2\right] &= E\left[f_1^2(X)\right] \\ &= \omega_1 \\ &= c, \end{aligned} \tag{61}$$

we must have that $X = -\sqrt{c}$, with probability $1/2$, and $X = \sqrt{c}$, with probability $1/2$.

For $-1 < q < 1$, since both $\{\alpha_n\}_{n\geq 0}$ and $\{\omega_n\}_{n\geq 1}$ are bounded sequences, the probability distribution of X, has a compact support. In fact, the distribution of X is known, namely, X is a continuous random variable, whose density function is the function:

$$f_{q,c}(x) \quad := \quad \frac{1}{\sqrt{c}} f_q(x/\sqrt{c}), \tag{62}$$

where:

$$f_q(x) \quad := \quad \frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{j=1}^{\infty} \left[\left(1+q^k\right)^2 - (1-q)x^2 q^k\right] \prod_{j=0}^{\infty} \left(1-q^{k+1}\right),$$

for $|x| < 4/(1-q)$.

For $q = 1$, X is a normally distributed random variable with mean zero and variance c.

Finally, for $q > 1$, the probability distribution of X cannot be determined by the moments, $E[X^n]$, $n \geq 1$, of X.

5. A Wick product characterization of q–Gaussian random variables

In the section we prove the main result of this paper.

Theorem 5.1 *Let X be a polynomially symmetric non–zero random variable, having finite moments of all orders. Then X is a q–Gaussian random variable if and only if there exists a sequence of real numbers $\{c_n\}_{n\geq 0}$ such that, for all f and g polynomials, we have:*

$$f(X) \diamond g(X) = \sum_{n=0}^{\infty} c_n \left[\left(a^-\right)^n f(X)\right] \cdot \left[\left(a^-\right)^n g(X)\right]. \tag{63}$$

If X is a q–Gaussian random variable with Szegö–Jacobi parameters $\{\omega_n\}_{n\geq 0}$, then, for all $n \geq 0$, we have:

$$c_n := (-1)^n \frac{q^{n(n-1)/2}}{\omega_n!}, \tag{64}$$

where:

$$\omega_n! = \begin{cases} 1 & \text{if} \quad n = 0 \\ \omega_1\omega_2\cdots\omega_n & \text{if} \quad n \geq 1 \end{cases}. \tag{65}$$

PROOF: ($\Leftarrow$) Let us suppose that formula (63) holds for some sequence $\{c_n\}_{n\geq 1}$ of real numbers. Choosing $f = g := f_0$, where $f_0 = 1$ (the constant polynomial equal to 1), since:

$$f_0 \diamond f_0 = f_0 \tag{66}$$

and

$$a^- 1 = 0, \tag{67}$$

we conclude from formula (63) that:

$$c_0 = 1. \tag{68}$$

For each natural number n, choosing $f := f_n$ and $g := f_1$, where $\{f_j\}_{j\geq 0}$ is the sequence of monic orthogonal polynomials generated by X, since

$$\left(a^-\right)^2 f_1(X) = 0, \tag{69}$$

formula (63) yields:

$$\begin{aligned} f_{n+1}(X) &= f_n(X) \diamond f_1(X) \\ &= \sum_{j=0}^{\infty} c_j \left[\left(a^-\right)^j f_n(X)\right] \cdot \left[\left(a^-\right)^j f_1(X)\right] \\ &= c_0 f_n(X) f_1(X) + c_1 a^- f_n(X) a^- f_1(X) \\ &= X f_n(X) + c_1 \omega_n f_{n-1}(X) \omega_1 f_0(X) \\ &= X f_n(X) + c_1 \omega_1 \omega_n f_{n-1}(X). \end{aligned} \tag{70}$$

Thus, we obtain:

$$X f_n(X) \quad = \quad f_{n+1}(X) - c_1 \omega_1 \omega_n f_{n-1}(X). \tag{71}$$

On the other hand, according to the definition of the Szegö–Jacobi parameters, since $\alpha_n = 0$, we have:

$$X f_n(X) \quad = \quad f_{n+1}(X) + \omega_n f_{n-1}(X). \tag{72}$$

We conclude from the last two relations that:

$$c_1 \quad = \quad -\frac{1}{\omega_1}, \tag{73}$$

where we used the fact that since $X \neq 0$, $\omega_1 \neq 0$. Finally, for each $n \geq 0$, choosing $f := f_n$ and $g := f_2$, relation (63) becomes:

$$\begin{aligned} & f_{n+2}(X) \qquad (74) \\ = \; & f_n(X) \diamond f_2(X) \\ = \; & c_0 f_n(X) f_2(X) + c_1 a^- f_n(X) a^- f_2(X) + c_2 \left(a^-\right)^2 f_n(X) \left(a^-\right)^2 f_2(X) \\ = \; & \left(X^2 - \omega_1\right) f_n(X) - \frac{1}{\omega_1} \omega_n f_{n-1}(X) \omega_2 X + c_2 \omega_n \omega_{n-1} f_{n-2}(X) \omega_2 \omega_1 1. \end{aligned}$$

Replacing $X f_n(X)$ by $f_{n+1}(X) + \omega_n f_{n-1}(X)$, and $X f_{n-1}(X)$ by $f_n(X) + \omega_{n-1} f_{n-2}(X)$, we obtain:

$$\begin{aligned} & f_{n+2}(X) \\ = \; & X \left[f_{n+1}(X) + \omega_n f_{n-1}(X)\right] - \omega_1 f_n(X) - \frac{\omega_2 \omega_n}{\omega_1} \left[f_n(X) + \omega_{n-1} f_{n-2}(X)\right] \\ & + c_2 \omega_1 \omega_2 \omega_{n-1} \omega_n f_{n-2}. \qquad (75) \end{aligned}$$

Replacing now $X f_{n+1}(X)$ by $f_{n+2}(X) + \omega_{n+1} f_n(X)$, and $X f_{n-1}(X)$ by $f_n(X) + \omega_{n-1} f_{n-2}(X)$, we obtain:

$$\begin{aligned} & f_{n+2}(X) \\ = \; & f_{n+2}(X) + \left(\omega_{n+1} + \omega_n - \omega_1 - \frac{\omega_2 \omega_n}{\omega_1}\right) f_n(X) \\ & + \left(\omega_{n-1} \omega_n - \frac{\omega_2 \omega_{n-1} \omega_n}{\omega_1} + c_2 \omega_1 \omega_2 \omega_{n-1} \omega_n\right) f_{n-2}(X). \qquad (76) \end{aligned}$$

Equating to zero the coefficient of $f_n(X)$ from the last relation, we obtain:

$$\omega_{n+1} - q \omega_n \quad = \quad c, \tag{77}$$

for all $n \geq 0$, where we define:

$$q \quad := \quad \frac{\omega_2 - \omega_1}{\omega_1} \tag{78}$$

and

$$c \;:=\; \omega_1. \tag{79}$$

Thus, we see that the ω–Szegö–Jacobi parameters obey the recursive relation (58), and so, X is a q–Gaussian random variable.
Moreover, equating the coefficient of f_{n-2} to zero, and dividing both sides by $\omega_n\omega_{n-1}$, we obtain:

$$c_2 \;=\; \frac{q}{\omega_1\omega_2}. \tag{80}$$

($\Rightarrow$) Let us suppose now that X is a q–Gaussian random variable. We would like to prove that for all f and g polynomials, we have:

$$f(X)\diamond g(X) \;=\; \sum_{i=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{\omega_n!}\left(a^-\right)^i f(X)\cdot\left(a^-\right)^i g(X). \tag{81}$$

It is enough to prove formula (81) for $f = f_m$ and $g = f_n$, for all m, $n \geq 0$.

We will prove it by induction on m.

For $m = 0$, since $a^- f_0(X) = 0$, formula (81) is clearly true for all $n \geq 0$.

For $m = 1$, for all $n \geq 0$, we have:

$$\begin{aligned}
&\sum_{i=0}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{\omega_i!}\left(a^-\right)^i f_1(X)\cdot\left(a^-\right)^i f_n(X)\\
=\;& f_1(X)f_n(X) - \frac{1}{\omega_1}a^- f_1(X)a^- f_n(X)\\
=\;& Xf_n(X) - \frac{1}{\omega_1}\omega_1 f_0(X)\cdot\omega_n f_{n-1}(X)\\
=\;& f_{n+1}(X) + \omega_n f_{n-1}(X) - \omega_n f_{n-1}(X)\\
=\;& f_{n+1}(X)\\
=\;& f_1(X)\diamond f_n(X).
\end{aligned}$$

Let us suppose now that formula (81) is true for $f = f_m$, where m is fixed, and for all $g = f_n$, $n \geq 0$. We would like to prove that it continues to be true for $f = f_{m+1}$, and all $g = f_n$, $n \geq 0$. We have the following Leibniz formula for q–commutators, see also [**?**]:

$$A^pB \;=\; \sum_{j=0}^{p-1} q^j A^{p-1-j}\left[A, B\right]_q A^j + q^p BA^p, \tag{82}$$

for all operators A and B, and p a natural number. Using this formula, we can write:

$$\begin{aligned}
&\sum_{i=0}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{\omega_i!}\left(a^-\right)^i f_{m+1}(X)\cdot\left(a^-\right)^i f_n(X)\\
= \;& f_{m+1}(X)f_n(X)+\sum_{i=1}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{\omega_i!}\left(a^-\right)^i a^+ f_m(X)\cdot\left(a^-\right)^i f_n(X)\\
= \;& \sum_{i=1}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{\omega_i!}\sum_{j=0}^{i-1} q^j\left(a^-\right)^{i-1-j}\left[a^-,a^+\right]_q (a^-)^j f_m(X)\cdot\left(a^-\right)^i f_n(X)\\
&+\sum_{i=0}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} q^i a^+\left(a^-\right)^i f_m(X)\cdot\left(a^-\right)^i f_n(X).
\end{aligned}$$

Replacing now $[a^-,\, a^+]_q$ by $\omega_1 I$, and a^+ in the last sum by $X-a^-$, we obtain:

$$\begin{aligned}
&\sum_{i=0}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{\omega_i!}\left(a^-\right)^i f_{m+1}(X)\cdot\left(a^-\right)^i f_n(X)\\
= \;& \sum_{i=1}^{\infty}\frac{(-1)^i q^{i(i-1)/2}}{\omega_i!}\sum_{j=0}^{i-1} q^j\left(a^-\right)^{i-1-j}\omega_1 I (a^-)^j f_m(X)\cdot\left(a^-\right)^i f_n(X)\\
&+\sum_{i=0}^{\infty}\frac{(-1)^i q^{i+i(i-1)/2}}{\omega_i!}\left(X-a^-\right)\left(a^-\right)^i f_m(X)\cdot\left(a^-\right)^i f_n(X).
\end{aligned}$$

Since:

$$\begin{aligned}
\sum_{j=0}^{i-1} q^j\left(a^-\right)^{i-1-j}\omega_1 I(a^-)^j f_m(X) &= \left[\sum_{j=0}^{i-1} q^j\right]\omega_1\left(a^-\right)^{i-1} f_m(X)\\
&= \frac{1-q^i}{1-q}\omega_1\left(a^-\right)^{i-1} f_m(X)\\
&= \omega_i\left(a^-\right)^{i-1} f_m(X),
\end{aligned}$$

and

$$i+\frac{i(i-1)}{2} = \frac{i(i+1)}{2},$$

we obtain:

$$
\begin{aligned}
& \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_{m+1}(X) \cdot \left(a^-\right)^i f_n(X) \\
= & \sum_{i=1}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \cdot \omega_i \left(a^-\right)^{i-1} f_m(X) \cdot \left(a^-\right)^i f_n(X) \\
& + \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i+1)/2}}{\omega_i!} \left[\left(X - a^-\right) \left(a^-\right)^i f_m(X)\right] \cdot \left(a^-\right)^i f_n(X) \\
= & \sum_{i=1}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_{i-1}!} \left(a^-\right)^{i-1} f_m(X) \cdot \left(a^-\right)^i f_n(X) \\
& - \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i+1)/2}}{\omega_i!} a^- \left(a^-\right)^i f_m(X) \cdot \left(a^-\right)^i f_n(X) \\
& + \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i+1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \cdot X \left(a^-\right)^i f_n(X).
\end{aligned}
$$

Making the change of variable $k = i - 1$ in the first sum, we obtain:

$$
\begin{aligned}
& \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_{m+1}(X) \cdot \left(a^-\right)^i f_n(X) \\
= & -\sum_{k=0}^{\infty} \frac{(-1)^k q^{(k+1)k/2}}{\omega_k!} \left(a^-\right)^k f_m(X) \cdot \left(a^-\right)^{k+1} f_n(X) \\
& -\sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i+1)/2}}{\omega_i!} \left(a^-\right)^{i+1} f_m(X) \cdot \left(a^-\right)^i f_n(X) \\
& +\sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i+1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \cdot \left(a^-\right)^{i+1} f_n(X) \\
& +\sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i+1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \cdot a^+ \left(a^-\right)^i f_n(X).
\end{aligned}
$$

We can see now that the first and third sums cancel. Thus we obtain:

$$
\begin{aligned}
& \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_{m+1}(X) \cdot \left(a^-\right)^i f_n(X) \\
= & -\sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i+1)/2}}{\omega_i!} \left(a^-\right)^{i+1} f_m(X) \cdot \left(a^-\right)^i f_n(X) \\
& +\sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i+1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \cdot a^+ \left(a^-\right)^i f_n(X).
\end{aligned}
$$

Making the change of variable $k = i + 1$ in the first sum, we obtain:

$$\begin{aligned}
& \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_{m+1}(X) \cdot \left(a^-\right)^i f_n(X) \\
= & \sum_{k=1}^{\infty} \frac{(-1)^k q^{(k-1)k/2}}{\omega_{k-1}!} \left(a^-\right)^k f_m(X) \cdot \left(a^-\right)^{k-1} f_n(X) \\
& + \sum_{i=1}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \cdot q^i a^+ \left(a^-\right)^i f_n(X) \\
& + f_m(X) f_{n+1}(X).
\end{aligned}$$

Since:

$$\begin{aligned}
& q^i a^+ \left(a^-\right)^i \\
= & q^i a^+ \left(a^-\right)^i - q^{i-1} a^- a^+ \left(a^-\right)^{i-1} \\
& + q^{i-1} a^- a^+ \left(a^-\right)^{i-1} - q^{i-2} \left(a^-\right)^2 a^+ \left(a^-\right)^{i-2} \\
& + \cdots \\
& + q \left(a^-\right)^{i-1} a^+ a^- - \left(a^-\right)^i a^+ \\
& + \left(a^-\right)^i a^+,
\end{aligned}$$

we have:

$$\begin{aligned}
q^i a^+ \left(a^-\right)^i &= -\sum_{j=0}^{i-1} q^j \left(a^-\right)^{i-1-j} \left[a^-, a^+\right]_q \left(a^-\right)^j + \left(a^-\right)^i a^+ \\
&= -\sum_{j=0}^{i-1} q^j \left(a^-\right)^{i-1-j} \omega_1 I \left(a^-\right)^j + \left(a^-\right)^i a^+ \\
&= -\frac{1-q^i}{1-q} \omega_1 \left(a^-\right)^{i-1} + \left(a^-\right)^i a^+ \\
&= -\omega_i \left(a^-\right)^{i-1} + \left(a^-\right)^i a^+.
\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}
& \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_{m+1}(X) \cdot \left(a^-\right)^i f_n(X) \\
= & \sum_{k=1}^{\infty} \frac{(-1)^k q^{(k-1)k/2}}{\omega_{k-1}!} \left(a^-\right)^k f_m(X) \cdot \left(a^-\right)^{k-1} f_n(X) \\
& + \sum_{i=1}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \cdot q^i a^+ \left(a^-\right)^i f_n(X) \\
& + f_m(X) f_{n+1}(X)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \frac{(-1)^i q^{(i-1)i/2}}{\omega_{i-1}!} \left(a^-\right)^i f_m(X) \cdot \left(a^-\right)^{i-1} f_n(X) \\
&\quad -\sum_{i=1}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \omega_i \left(a^-\right)^i f_m(X) \cdot \left(a^-\right)^{i-1} f_n(X) \\
&\quad +\sum_{i=1}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \left(a^-\right)^i a^+ f_n(X) \\
&\quad + f_m(X) f_{n+1}(X) \\
&= \sum_{i=1}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \left(a^-\right)^i f_{n+1}(X) + f_m(X) f_{n+1}(X) \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \left(a^-\right)^i f_{n+1}(X).
\end{aligned}
$$

According to the induction hypothesis, we have:

$$
\begin{aligned}
& \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_{m+1}(X) \cdot \left(a^-\right)^i f_n(X) \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i q^{i(i-1)/2}}{\omega_i!} \left(a^-\right)^i f_m(X) \left(a^-\right)^i f_{n+1}(X) \\
&= f_m(X) \diamond f_{n+1}(X) \\
&= f_{m+n+1}(X) \\
&= f_{m+1}(X) \diamond f_n(X).
\end{aligned}
$$

Our proof is now complete.

We also have the following characterization theorem:

Theorem 5.2 *Let X be a polynomially symmetric non–zero random variable, having finite moments of all orders. Then X is a q–Gaussian random variable if and only if there exists a sequence of real numbers $\{d_n\}_{n \geq 0}$ such that, for all f and g polynomials, we have:*

$$
f(X) \cdot g(X) = \sum_{n=0}^{\infty} d_n \left[\left(a^-\right)^n f(X)\right] \diamond \left[\left(a^-\right)^n g(X)\right]. \tag{83}
$$

If X is a q–Gaussian random variable, then:

$$
d_n = \frac{1}{\omega_n!}, \tag{84}
$$

for all $n \geq 0$.

The proof is similar to the proof of the previous theorem. The essential property used in the proof of the previous theorem is:

$$
[X \cdot f(X)] \cdot g(X) = f(x) \cdot [X \cdot g(X)]. \tag{85}
$$

The important property used in the proof of the last theorem is:

$$[X \diamond f(X)] \diamond g(X) \quad = \quad f(x) \diamond [X \diamond g(X)] \,. \tag{86}$$

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