Parametric Continuity of Choquet and Sugeno Integrals

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Abstract

We consider a fixed probability μ. For any $\lambda \in (-1, \infty)$, we denote by $m(\lambda, \mu)$ the λ-Sugeno measure generated by μ. For a fixed measurable set A, the continuity of the map $\lambda \longmapsto \int_A f dm(\lambda, \mu)$ is proved, where the integral is either Choquet or Sugeno. The asymptotic (marginal) cases $\lambda = -1$ and $\lambda = \infty$ are studied too. Finally, some concrete computations illustrate the previous theory.

Keywords: (classical) measure, monotone measure, λ-measure, Choquet integral, Sugeno integral
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1. Introduction

Classical measure theory and classical integration theory are based on the concept of additivity (or, which is more, countable additivity). Recently, new necessities (theoretical and practical) imposed the study of (possibly) non additive measures. These measures and the associated integrals play an increasing role in the description of all kind of phenomena and are intensively studied now.

A major role in the history of generalized (i.e. monotone and possibly non additive) measures was played by the Japanese scholar M. Sugeno who formally introduced them (calling them fuzzy measures), together with the Sugeno integral (called by him fuzzy integral) in his doctoral thesis [5]. Trying to study non-additivity in the same thesis, he introduced the concept of λ-additivity (λ-rule) which averred to be extremely important, raising extremely many further developments. Among these developments, we can nominate the important fact that any λ-measure is generated by a classical measure, this fact being proved by Z. Wang in [6].

Associated with non additive measures are Choquet and Sugeno integrals. The Choquet integral (formally definitivated in [4]) generalizes the classical abstract Lebesgue integral. The Sugeno integral (already mentioned) modelates some phenomena. Both integrals serve to aggregate data.

In the present paper we study the continuity (with respect to the parameter λ) of the Choquet and Sugeno integrals $\int_A f dm(\lambda, \mu)$. Here, the set A is fixed, the probability μ is fixed, the parameter λ runs over $(-1, \infty)$ and $m(\lambda, \mu)$ is the λ-Sugeno measure generated by μ. The asymptotic behaviour (when λ tends to -1 or to ∞) is studied too. In order to make facts more concrete, we close our paper with some computations.

Remark. The present paper is an extended version of the communication (with the same title) presented by the author at the ISAAC 9-th Congress, Krakow 2013.

2. **Preliminary facts**

As usual: $\mathbb{R}$ = the real numbers, $\mathbb{R}_+ = [0, \infty)$, $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \bigcup \{\infty\}$ and $\mathbb{N} = \{1, 2, \ldots, n, \ldots\}$. For $a, b \in \mathbb{R}$, $a \wedge b = min\,\{a, b\}$.

If T is a non empty set, $\mathcal{P}(T)$ will be the set of all sets $A \subset T$. For any $A \in \mathcal{P}(T)$, $\chi_A : T \to \mathbb{R}$ is the characteristic (indicator) function of A: $\chi_A(t) = 0$ if $t \notin A$ and $\chi_A(t) = 1$ if $t \in A$. If X is a non empty set and $(x_n)_n$ is a sequence with all terms $x_n \in X$, we shall write $(x_n)_n \subset X$.

For any $f : T \to \mathbb{R}_+$ and any $\alpha \in \mathbb{R}_+$, we define the level set

$$F_\alpha(f) = F_\alpha = \{t \in T | f(t) \geq \alpha\}$$

(we write F_α, omitting the specification of f, if f is understood).

A measurable space is a couple $(T, \mathcal{T})$ where T is a non empty set and $\mathcal{T} \subset \mathcal{P}(T)$ is a σ-algebra. A function $f : T \to \mathbb{R}$ is called $\mathcal{T}$-measurable (measurable if $\mathcal{T}$ is understood) if $f^{-1}(B) \in \mathcal{T}$ for any Borel set $B \subset \mathbb{R}$. Same definition in case $f : T \to \mathbb{R}_+$.

A classical measure space $(T, \mathcal{T}, \mu)$ is a triple where T is a non empty set, $\mathcal{T} \subset \mathcal{P}(T)$ is a σ-algebra and $\mu : \mathcal{T} \to \mathbb{R}_+$ is a (finite) measure. We shall always assume that μ is not null (i.e. $\mu(T) > 0$) and complete (i.e. if $A \in \mathcal{T}$, $\mu(A) = 0$ and $B \subset A$, one has $B \in \mathcal{T}$). Practically, we shall work only with probabilistic measures, (or probabilities) μ, i.e. we shall assume that $\mu(T) = 1$. In this case, $(T, \mathcal{T}, \mu)$ will be called a probabilistic space. Within the framework of a measure space $(T, \mathcal{T}, \mu)$, a function $f : T \to \mathbb{R}$ (or $f : T \to \mathbb{R}_+$) which is $\mathcal{T}$-measurable is often called μ-measurable and we shall call it measurable function. It is seen that for any measurable $f : T \to \mathbb{R}_+$ and for any $\alpha \in \mathbb{R}_+$, one has $F_\alpha \in \mathcal{T}$ (as always, we write F_α instead of $F_\alpha(f)$).

Now, we can recall the following:

Theorem. *For any measurable function $f : T \to \mathbb{R}_+$ and any $A \in \mathcal{T}$, one has*

$$\int_A f d\mu = \int_0^\infty \mu(F_\alpha \bigcap A) d\alpha.$$

The last term means $\int \varphi dL$, where $\varphi : T \to \mathbb{R}_+$ acts via $\varphi(\alpha) = \mu(F_\alpha \bigcap A)$ and L is the Lebesgue measure on $\mathbb{R}_+$.

This formula is the basis of the theory of Choquet integral (see later) which generalizes the usual (abstract) Lebesgue integral.

Also within this framework, let us recall that the set of all μ-integrable functions $f : T \to \mathbb{R}$ is a vector space denoted by $\mathcal{L}^1(\mu)$. This space is seminormed with the usual seminorm $f \longmapsto ||f||_1 = \int |f| d\mu$. On this space we have the linear and continuous functional $H : \mathcal{L}^1(\mu) \to \mathbb{R}$ given via

$$H(f) = \int f d\mu.$$

We pass to the theory of generalized measures and integrals.

Let $(T, \mathcal{T})$ be a measurable space. We say that a function $m : \mathcal{T} \to \overline{\mathbb{R}}_+$ is a monotone measure if $m(\emptyset) = 0$ and $m(A) \leq m(B)$, whenever A, B are in $\mathcal{T}$ and $A \subset B$. We shall always assume that $m(T) > 0$ (i.e. m is not identically null). We say that is finite if $m(A) < \infty$ for any $A \in \mathcal{T}$ (i.e. $m(T) < \infty$).

The monotone measure m is called continuous if: a) For any increasing sequence $(A_n)_n \subset \mathcal{T}$ one has $m(\bigcup_n A_n) = sup_n(A_n)$, b) For any decreasing sequence $(A_n)_n \subset \mathcal{T}$ such that there exists n_0 with $m(A_{n_0}) < \infty$, one has $m(\bigcap_n A_n) = inf_n m(A_n)$. Consequently, if m is finite, to say that m is continuous means to say that $m(lim_n A_n) = lim_n m(A_n)$ for any monotone sequence $(A_n)_n \subset \mathcal{T}$. Any classical finite measure is continuous. The Lebesgue measure on any interval is continuous.

Now, we shall present the Choquet integral (which extends the classical abstract Lebesgue integral) and the Sugeno integral (for positive measurable functions).

Let $(T, \mathcal{T})$ be a measurable space, $f : T \to \mathbb{R}_+$ a measurable function and $m : \mathcal{T} \to \mathbb{R}_+$ a finite monotone measure.

Definition (Choquet Integral). *For any $A \in \mathcal{T}$, the Choquet integral of f with respect to m on A is*

$$(C) \int_A f dm = \int_0^\infty m(F_\alpha \bigcap A) d\alpha \leq \infty.$$

In case $A = T$, we write only

$$(C) \int f dm = \int_0^\infty m(F_\alpha) d\alpha \leq \infty$$

(this is the Choquet integral of f with respect to m).

In case $\int f dm < \infty$, we say that f is Choquet integrable with respect to m.

Definition (Sugeno Integral). *For any $A \in \mathcal{T}$, the Sugeno integral of f with respect to m on A is*

$$(S) \int_A f dm = \sup_{\alpha \in \mathbb{R}_+} (\alpha \wedge m(F_\alpha \bigcap A)) \leq m(T).$$

In case $A = T$, we write only

$$(S) \int f dm = \sup_{\alpha \in \mathbb{R}_+} (\alpha \wedge m(F_\alpha)) \leq m(T)$$

(this is the Sugeno integral of f with respect to m).

Remarks:

- For any $A \in \mathcal{T}$, one has

$$(C)\int_A f dm = (C)\int f\chi_A dm \le \int f dm$$

$$(S)\int f dm = (S)\int f\chi_A dm \le \int f dm.$$

- In case $m(T) \le 1$, we have

$$(S)\int f dm = \sup_{\alpha\in[0,1]} (\alpha \wedge m(F_\alpha))$$

  because for $\alpha > 1$, one has $\alpha \wedge m(F_\alpha) = m(F_\alpha) \le m(F_1) = 1 \wedge m(F_1)$.

We pass to a special type of monotone measures. For our purposes, from now on we shall work only for normalized monotone measures, i.e. monotone measures m with total mass $m(T) = 1$ (generalized probabilities). We shall call them measures for simplicity.

Let $(T, \mathcal{T})$ be a measurable space and $m : \mathcal{T} \to \mathbb{R}_+$ be a (normalized monotone) measure.

Definition. *Let $\lambda \in (-1, \infty)$. We say that m satisfies the λ-rule (or is λ-additive) if*

$$M(E\bigcup F) = m(E) + m(F) + \lambda m(E)m(F)$$

whenever E, F are in $\mathcal{T}$ and $E\bigcap F = \emptyset$.

One can prove that m satisfies the λ-rule if and only if it satisfies the finite λ-rule, i.e., for any $2 \le n \in \mathbb{N}$ and any mutually disjoit sets $E_1, E_2, \ldots, E_n in \mathcal{T}$ one has

$$m(\bigcup_{i=1}^n E_i) = \begin{cases} \sum\limits_{i=1}^n m(E_i) & \text{if } \lambda = 0 \\ \frac{1}{\lambda}(\prod\limits_{i=1}^n (1+\lambda m(E_i)) - 1) & \text{if } \lambda \ne 0 \end{cases}.$$

So 0-rule (or finite 0-rule) means additivity (or finite additivity).

A stronger property is given in the following:

Definition. *Again let $\lambda \in (-1, \infty)$. We say that m satisfies the σ-λ-rule (or, is a λ-measure) if, for any sequence $(E_i)_i$ of mutually disjoint sets $E_i \in \mathcal{T}$, one has*

$$m(\bigcup_{i=1}^\infty E_i) = \begin{cases} \sum\limits_{i=1}^\infty m(E_i) & \text{if } \lambda = 0 \\ \frac{1}{\lambda}(\prod\limits_{i=1}^\infty (1+\lambda m(E_i)) - 1) & \text{if } \lambda \ne 0 \end{cases}.$$

One can prove that σ-λ-rule implies (finite) λ-rule (the converse implication is not true). Clearly, σ-0-rule means countable additivity.

Definition (M. Sugeno, 1974). *If m satisfies the σ-λ-rule, we say that m is a λ-Sugeno measure (we call m a Sugeno measure if there exists λ such that m is a λ-Sugeno measure).*

Remark. M. Sugeno, in his doctoral thesis [5], called ***fuzzy measures*** the Sugeno measures.

Definition. *Let $(T, \mathcal{T}, \mu)$ be a probabilistic space and let $\lambda \in (-1, \infty)$. The λ-Sugeno measure generated by μ is $m(\lambda, \mu) = m(\lambda) : \mathcal{T} \to \mathbb{R}_+$ defined via*

$$m(\lambda)(A) = \begin{cases} \frac{(\lambda+1)^{\mu(A)}-1}{\lambda}, & \text{if } \lambda \neq 0 \\ \mu(A), & \text{if } \lambda = 0 \end{cases}.$$

Remark. We omit to write μ and write only $m(\lambda)$ if μ is understood. One can prove that $m(\lambda)$ is a λ-Sugeno measure.

Theorem (Z. Wang, 1981, see [6]). *Let $(T, \mathcal{T})$ be a measurable space and let $\lambda \in (-1, \infty)$.*

There exists a bijective correspondence between the probabilities $\mu : \mathcal{T} \to \mathbb{R}_+$ and the λ-Sugeno measures $m : \mathcal{T} \to \mathbb{R}_+$ given as follows:

$$\mu \longmapsto m(\lambda, \mu) = m(\lambda).$$

The basic theoretical material for this paper is the monograph [7]. For general (classical) measure theory see [2] and [3]. For the Choquet integral, one can consult supplementarily [1] and [4]. For the Sugeno integral, one can consult supplementarily [5].

Caution. In the next part of this paper, we shall consider a fixed probabilistic space $(T, \mathcal{T}, \mu)$ and a fixed positive measurable function $f : T \to \mathbb{R}_+$ (which will be, sometimes, μ-integrable). All the notations and constructions will refer to μ and f. For instance, if $\alpha \in \mathbb{R}_+$, we shall consider the level set $F_\alpha = F_\alpha(f)$. If $\lambda \in (-1, \infty)$, we shall consider the λ-Sugeno measure $m(\lambda) = m(\lambda, \mu)$.

3. Results

3.1 *Normal Parametric Continuity*

Lemma 1. *Let $1 \leq a \leq 1$. We define the function $\varphi : (-1, \infty) \to \mathbb{R}_+$ via*

$$\varphi(\lambda) = \begin{cases} \frac{(\lambda+1)^a-1}{\lambda}, & \text{if } \lambda \neq 0 \\ a, & \text{if } \lambda = 0 \end{cases}.$$

Then φ is continuously differentiable. In case $a = 0$ or $a = 1$, φ is constant. In case $0 < a < 1$ one has $\varphi'(\lambda) < 0$ for any $\lambda \in (-1, \infty)$. In all cases φ is decreasing. In case $0 < a < 1$, one has $0 < \varphi(\lambda) < 1$ for any $\lambda \in (-1, \infty)$.

Proof. It is readily seen that φ is continuous. For any $\lambda \neq 0$, one has

$$\varphi'(\lambda) = \frac{(\lambda+1)^{a-1}(a\lambda - \lambda - 1) + 1}{\lambda^2} = \frac{A(\lambda)}{B(\lambda)} \tag{1}$$

and

$$\frac{A'(\lambda)}{B'(\lambda)} = \frac{a(a-1)}{2}(\lambda+1)^{a-2} \underset{\lambda \to 0}{\longrightarrow} \frac{a(a-1)}{2}$$

hence

$$\lim_{\lambda \to 0} \varphi'(\lambda) = \frac{a(a-1)}{2}. \tag{2}$$

The function φ is continuous at 0, differentiable on $(-1,\infty)\setminus\{0\}$ and satisfies (2). Using a corollary of Lagrange′s theorem, we get

$$\varphi''(0) = \lim_{\lambda \to 0} \varphi'(\lambda) = \frac{a(a-1)}{2} \tag{3}$$

If $a = 0$, one has $\varphi \equiv 0$ and if $a = 1$, one has $\varphi \equiv 1$.
Let us assume $0 < a < 1$. Writing (see(1))

$$A(\lambda) = (\lambda+1)^{a-1}(a\lambda - \lambda - 1) + 1$$

one has

$$A'(\lambda) = a(a-1)\lambda(\lambda+1)^{a-2}$$

and $A'(\lambda) > 0$ for $-1 < \lambda < 0$, $A'(\lambda) < 0$ for $\lambda > 0$. This shows (because $A(0) = 0$) that $A(\lambda) < 0$ for all $\lambda \in (-1,\infty)\setminus\{0\}$. Hence $\varphi'(\lambda) < 0$ for any $\lambda \in (-1,\infty)$ (see (3)).

The last property follows taking the limits at -1 and ∞ and using the monotonicity.

□

Theorem 2. *Assume $f : T \to \mathbb{R}_+$ is μ-integrable. We have the following results:*

1. *The function f is Choquet integrable with respect to $m(\lambda)$, for any $\lambda \in (-1,\infty)$.*
2. *For any $A \in \mathcal{T}$, the function $V : (-1,\infty) \to \mathbb{R}$ given via*

$$V(\lambda) = (C)\int_A f dm(\lambda)$$

is continuous.

Proof.

1. The result is obvious for $\lambda = 0$. It is also seen that the result is obvious in case there exists $\alpha_0 > 0$ such that $\mu(F_{\alpha_0}) = 0$. Let us consider $\lambda \neq 0$ and $\mu(F_\alpha) > 0$ for any $\alpha > 0$. One must prove that

$$\int_0^\infty |(\lambda + 1)^{\mu(F_\alpha)-1}| d\alpha < \infty.$$

The function $\alpha \longmapsto (\lambda + 1)^{\mu(F_\alpha)} - 1$ is monotone, hence it is Borel measurable and its values are either all positive or all negative. We can apply Cauchy$'$s integral criterion: it suffices to show that the series

$$\sum_{n=1}^{\infty} ((\lambda + 1)^{\mu(F_\alpha)} - 1) \tag{4}$$

is convergent. To prove this, we notice that, because of the hypothesis (integrability of f) which says that

$$\int f d\mu = \int_0^\infty \mu(F_\alpha) d\alpha < \infty$$

the same criterion tells us that the series

$$\sum_{n=1}^{\infty} \mu(F_n)$$

is convergent. We have $\bigcap\limits_{n-1}^{\infty} F_n = \emptyset$, hence $\lim\limits_n \mu(F_n) = 0$ and it follows that

$$\lim_n \frac{(\lambda + 1)^{\mu(F_n)} - 1}{\mu(F_n)} = ln(\lambda + 1) \neq 0.$$

A comparison criterion implies that the series (4) is convergent too.

2. One can work only for $A = T$, because, for any $\lambda \in (-1, \infty)$ and any $A \in \mathcal{T}$, one has

$$(C) \int_A f dm(\lambda) = (C) \int f \chi_A dm(\lambda)$$

and we use $f\chi_A$ instead of f.

We must prove that, for any $\lambda_0 \in (-1, \infty)$, one has

$$\lim_{\lambda \to \lambda_0} \int f dm(\lambda) = (C) \int f dm(\lambda_0). \tag{5}$$

In order to prove (5), we consider the measure space $(\mathbb{R}_+, \mathcal{L}, L)$ where $\mathcal{L}$ = the Lebesgue measurable subsets of $\mathbb{R}_+$ and L is the Lebesgue measure on $\mathbb{R}_+$. Using point 1, we have, for any $\lambda \in (-1, \infty)$, the function $h(\lambda) \in \mathcal{L}^1(L)$, given via

$$h(\lambda)(\alpha) = m(\lambda)(F_\alpha).$$

The idea is to prove that the function $W : (-1, \infty) \to \mathcal{L}^1(L)$, given via $W(\lambda) = h(\lambda)$, is continuous, i.e. is continuous at the arbitrary point $\lambda_0 \in (-1, \infty)$, which means that

$$\lim_{\lambda \to \lambda_0} ||h(\lambda) - h(\lambda_0||_1 = 0. \tag{6}$$

Indeed, accepting the validity of (6), we consider the linear and continuous functional $H : \mathcal{L}^1(L) \to \mathbb{R}$, given via

$$H(h) = \int h dL = \int_0^\infty h(\alpha) d\alpha$$

and we obtain the function $H \circ W : (-1, \infty) \to \mathbb{R}$ which is continuous at λ_0, exactly translating what (5) is telling us.

The proof of (6) will be done separately for $\lambda_0 \neq 0$ and $\lambda_0 = 0$.

- I. **Case** $\lambda_0 \neq 0$

  Let $\varepsilon > 0$. Write

  $$A = \int f d\mu = \int_0^\infty \mu(F_\alpha) d\alpha.$$

  It is possible to find $0 < \delta < 1 + \lambda_0$ such that $0 \notin [\lambda_0 - \delta, \lambda_0 + \delta]$ and having the following properties:

  $$|\lambda - \lambda_0| < \delta \Longrightarrow \frac{1}{|\lambda\lambda_0|} = \frac{1}{\lambda\lambda_0} < \frac{2}{\lambda_0^2} \tag{7}$$

  $$\delta \frac{2A}{\lambda_0^2} \exp(|\ln(\lambda_0 + 1)|) < \frac{\varepsilon}{2} \tag{8}$$

  $$|t - t_0| < \delta \Longrightarrow |ln(t + 1)| < 2|ln(\lambda_0 + 1)| \tag{9}$$

  $$|t - \lambda_0| < \delta \Longrightarrow \left|\frac{1}{t+1}\right| = \frac{1}{t+1} < \frac{2}{|\lambda_0 + 1|} = \frac{2}{\lambda_0 + 1} \tag{10}$$

  $$|t - \lambda_0| < \delta \Longrightarrow \left|\frac{1}{t}\right| < \left|\frac{2}{\lambda_0}\right| \tag{11}$$

  $$\delta \frac{4A}{|\lambda_0|(\lambda_0 + 1)} \exp(2|\ln(\lambda_0 + 1)|) < \frac{\varepsilon}{2} \tag{12}$$

For this δ, we shall prove that, if $|\lambda - \lambda_0| < \delta$, one has $||h(\lambda) - h(\lambda_0)|| < \varepsilon$, i.e.

$$\int \left| \frac{(\lambda+1)^{\mu(F_\alpha)} - 1}{\lambda} - \frac{(\lambda_0+1)^{\mu(F_\alpha)} - 1}{\lambda_0} \right| d\alpha < \varepsilon \tag{13}$$

which is the final goal of this point I.
To this end, we shall consider the Mac Laurin expansion (with convergence radius equal to ∞) of the functions

$$x \longmapsto (\lambda+1)^x = \exp(xln(\lambda+1))$$

$$x \longmapsto (\lambda_0+1)^x = \exp(xln(\lambda_0+1))$$

for $x = \mu(F_\alpha)$, where $\alpha \in \mathbb{R}_+$ is arbitrary:

$$(\lambda+1)^{\mu(F_\alpha)} = 1 + \sum_{n=1}^{\infty} \frac{(ln(\lambda+1))^n}{n!}(\mu(F_\alpha))^n$$

$$(\lambda_0+1)^{\mu(F_\alpha)} = 1 + \sum_{n=1}^{\infty} \frac{(ln(\lambda_0+1))^n}{n!}(\mu(F_\alpha))^n.$$

This gives

$$\begin{aligned} &\left| \frac{(\lambda+1)^{\mu(F_\alpha)}-1}{\lambda} - \frac{(\lambda_0+1)^{\mu(F_\alpha)}-1}{\lambda_0} \right| \le \\ &\le \sum_{n=1}^{\infty} \frac{(\mu(F_\alpha))^n}{n!} \left| \frac{(ln(\lambda+1))^n}{\lambda} - \frac{(ln(\lambda_0+1))^n}{\lambda_0} \right| \le \\ &\le \sum_{n=1}^{\infty} \frac{\mu(F_\alpha)}{n!} \left| \frac{(ln(\lambda+1))^n}{\lambda} - \frac{(ln(\lambda_0+1))^n}{\lambda_0} \right|. \end{aligned}$$

Consequently

$$\begin{aligned} &\int \left| \frac{(\lambda+1)^{\mu(F_\alpha)}-1}{\lambda} - \frac{(\lambda_0+1)^{\mu(F_\alpha)}-1}{\lambda_0} \right| d\alpha \le \\ &\le A \sum_{n=1}^{\infty} \left| \frac{(ln(\lambda+1))^n}{\lambda} - \frac{(ln(\lambda_0+1))^n}{\lambda_0} \right| \le \\ &\le A \sum_{n=1}^{\infty} \frac{1}{n!} \left| \frac{\lambda_0(ln(\lambda+1))^n - \lambda_0(ln(\lambda_0+1))^n + \lambda_0(ln(\lambda_0+1))^n - \lambda(ln(\lambda_0+1))^n}{\lambda\lambda_0} \right| \le \end{aligned}$$

$$\begin{aligned} &\le A \sum_{n=1}^{\infty} \frac{1}{n!} \left| \frac{(ln(\lambda+1))^n - (ln(\lambda_0+1))^n}{\lambda} \right| + \\ &+ A \sum_{n=1}^{\infty} \left| \frac{\lambda-\lambda_0}{\lambda\lambda_0} \right| |ln(\lambda_0+1)|^n = U + V \end{aligned} \tag{14}$$

Lagrange′s theorem gives

$$(ln(\lambda+1))^n - (ln(\lambda_0+1))^n = (\lambda-\lambda_0)nln(t_n+1)^{n-1}\frac{1}{t_n+1}$$

where $t_n \in (\lambda_0 - \delta, \lambda_0 + \delta)$, hence (9), (10) and (11) give

$$|ln(t_n + 1)| < 2|ln(\lambda_0 + 1)|,$$

$$\frac{1}{t_n + 1} = \frac{1}{|t_n + 1|} < \frac{2}{\lambda_0 + 1}$$

and

$$\left|\frac{1}{\lambda}\right| < \left|\frac{2}{\lambda_0}\right|.$$

Consequently, using (12):

$$\begin{aligned} U \leq A \sum_{n=1}^{\infty} \tfrac{1}{n!} \cdot \tfrac{2}{|\lambda_0|} \cdot n \cdot 2^{n-1} |ln(\lambda_0 + 1)|^{n-1} \tfrac{2}{\lambda_0+1} |\lambda - \lambda_0| \leq \\ \leq \tfrac{4A\delta}{|\lambda_0|(\lambda_0+1)} \sum_{n=1}^{\infty} \tfrac{2^{n-1}|ln(\lambda_0+1)|^{n-1}}{(n-1)!} = \\ = \delta \frac{4A}{|\lambda_0|(\lambda_0 + 1)} \exp(2|ln(\lambda_0 + 1)|) < \frac{\varepsilon}{2} \end{aligned} \tag{15}$$

On the other hand, (7) gives

$$\frac{1}{|\lambda\lambda_0|} < \frac{2}{\lambda_0^2}.$$

Consequently, using (8):

$$\begin{aligned} V \leq A \tfrac{2}{\lambda_0^2} \delta \sum_{n=1}^{\infty} \tfrac{1}{n!} |ln(\lambda_0 + 1)|^n \leq \\ \leq \delta \tfrac{2A}{\lambda_0^2} \sum_{n=0}^{\infty} \tfrac{|ln(\lambda_0+1)|^n}{n!} = \\ = \delta \tfrac{2A}{\lambda_0^2} \exp(|ln(\lambda_0 + 1)|) < \tfrac{\varepsilon}{2}. \end{aligned} \tag{16}$$

From (14), (15) and (16), we get (13).

- **II. Case** $\lambda_0 = 0$

For any $\alpha \in \mathbb{R}_+$ and any $\lambda \in (-1, 1)$ one has (binomial series):

$$(\lambda + 1)^{\mu(F_\alpha)} = 1 + \sum_{p=1}^{\infty} \frac{\mu(F_\alpha)(\mu(F_\alpha) - 1) \dots (\mu(F_\alpha) - p + 1)}{p!} \lambda^{p-1}.$$

Consequently, if $0 \neq \lambda \in (-1, 1)$:

$$\begin{aligned} m(\lambda)(F_\alpha) - m(0)(F_\alpha) = m(\lambda)(F_\alpha) - \mu(F_\alpha) = \\ = \tfrac{(\lambda+1)^{\mu(F_\alpha)} - 1}{\lambda} - \mu(F_\alpha) = \\ = \sum_{p=2}^{\infty} \tfrac{\mu(F_\alpha)(\mu(F_\alpha)-1) \dots (\mu(F_\alpha)-p+1)}{p!} \lambda^{p-1} \end{aligned}$$

For any $k = 1, 2, \ldots, p-1$, one has

$$|\mu(F_\alpha) - k| \le k$$

and this implies, for any $p \ge 2$ and any $\alpha \in \mathbb{R}_+$ (and, of course, any $0 \ne \lambda \in (-1, 1)$):

$$|u_p(\alpha)| \le \mu(F_\alpha)\frac{(p-1)!}{p!}|\lambda|^{p-1} = \mu(F_\alpha)\frac{|\lambda|^{p-1}}{p}$$

where $u_p : \mathbb{R}_+ \to \mathbb{R}$ is given via

$$u_p(\alpha) = \frac{\mu(F_\alpha)(\mu(F_\alpha) - 1)\ldots(\mu(F_\alpha) - p + 1)}{p!}\lambda^{p-1}.$$

In view of the monotonicity of the function $\alpha \longmapsto \mu(F_\alpha)$, it is seen that all u_p are (as function of α) Borel measurable and we have, for any $\alpha \in \mathbb{R}_+$:

$$|m(\lambda)(F_\alpha) - \mu(F_\alpha)| \le \mu(F_\alpha)\sum_{p=2}^{\infty}|\lambda|^{p-1} = \mu(F_\alpha)\frac{|\lambda|}{1 - |\lambda|}.$$

Consequently, for $0 \ne \lambda \in (-1, 1)$:

$$\begin{aligned} ||h(\lambda) - h(0)||_1 = \textstyle\int_0^\infty |m(\lambda)(F_\alpha) - \mu(F_\alpha)|d\alpha \le \\ \le \tfrac{|\lambda|}{1-|\lambda|}\textstyle\int_0^\infty \mu(F_\alpha)d\alpha = \int f d\mu \underset{\lambda}{\longrightarrow} 0 \end{aligned}$$

and the proof is finished.

□

Theorem 3. *Assume $f : T \to \mathbb{R}_+$ is measurable. For any $A \in \mathcal{T}$, the function $V : (-1, \infty) \to \mathbb{R}$, given via*

$$V(\lambda) = (S)\int_A dm(\lambda)$$

is continuous.

Proof. Again we can work only for $A = T$, because, for any $A \in \mathcal{T}$, one has

$$(S)\int_A dm(\lambda) = (S)\int f\chi_A dm(\lambda)$$

and we can use $f\chi_A$ instead of f.

Let $\lambda_0 \in (-1, \infty)$ be arbitrarily taken and let us prove that V is continuous at λ_0. To this end, we consider a strictly monotone sequence of numbers $\lambda_0 \ne \lambda_n \underset{n}{\longrightarrow} \lambda_0 in(-1, \infty)$ and let us construct the functions

$$h_n : \mathbb{R}_+ \to \mathbb{R}_+,\ h_n(\alpha) = m(\lambda_n)(F_\alpha)$$

$$h : \mathbb{R}_+ \to \mathbb{R}_+,\ h(\alpha) = m(\lambda_0)(F_\alpha).$$

According to the Transformation Theorem (Theorem 9.13 from [7]), we have the equalities:

$$(S)\int f dm(\lambda_n) = (S)\int_0^\infty h_n(\alpha)d\alpha = (S)\int h_n dL$$

$$(S)\int f dm(\lambda_0) = (S)\int_0^\infty h(\alpha)d\alpha = (S)\int h dL$$

where L if the Lebesgue measure on $\mathbb{R}_+$.

Consequently, we must prove that

$$\lim_n (S)\int_0^\infty h_n(\alpha)d\alpha = (S)\int_0^\infty h(\alpha)d\alpha \tag{17}$$

(equal unilateral limits).

According to Lemma 1, the sequence $(h_n)_n$ is monotone and has pointwise limit h.

A First, let us consider the case when $\lambda_n > \lambda_0$ (strictly decreasing sequence). In this case, Lemma 1 tells us that the sequence $(h_n)_n$ is increasing and has pointwise limit h. According to Theorem 9.5 from [7], equality (17) is true, because the Lebesgue measure is continuous.

B Now, let us consider the case when $\lambda_n < \lambda_0$ (strictly increasing sequence). In this case, Lemma 1 tells us that the sequence$(h_n)_n$ is decreasing and has pointwise limit h.

Write

$$C = (S)\int_0^\infty h(\alpha)d\alpha = (S)\int h dL.$$

First possibility: $C = 0$.
This means:

$$\sup_{\alpha\in\mathbb{R}_+} \alpha \wedge m(\lambda_0)F_\alpha) = 0,$$

i.e. $m(\lambda_0)(F_\alpha) = 0$. In other words: for any $\alpha > 0$, one has

$$\lim_n m(\lambda_n)(F_\alpha) = \inf_n m(\lambda_n)(F_\alpha).$$

Let accept the (absurd) fact that

$$\lim_n (S)\int f dm(\lambda_n) = \inf_n (S)\int f dm(\lambda_n) > 0 = (S)\int f dm(\lambda_0).$$

Then we can find a number $\alpha_0 > 0$ such that, for any n one has

$$(S)\int f dm(\lambda_n) = \sup_{\alpha\in\mathbb{R}_+} (\alpha \wedge m(\lambda_n)(F_\alpha)) \geq \alpha_0.$$

Fix $0 < \varepsilon < \alpha_0$, hence $\alpha_0 - \varepsilon < \sup_{\alpha \in \mathbb{R}_+} (\alpha \wedge m(\lambda_n)(F_\alpha))$ for any n. Hence, for any n, one can find a number $\alpha_n > 0$ such that $\alpha_n \wedge m(\lambda_n)(F_{\alpha_n}) > \alpha_0 - \varepsilon$. Clearly $\alpha_n > \alpha_0 - \varepsilon$, hence, for any n:

$$m(\lambda_n)(F_{\alpha_n}) \leq m(\lambda_n))F_{\alpha_0 - \varepsilon}) \underset{n}{\longrightarrow} 0.$$

contradicting the fact that, for any n:

$$\alpha_0 - \varepsilon < \alpha_n \wedge m(\lambda_n)(F_{\alpha_n}) \leq m(\lambda_n)(F_{\alpha_n}).$$

Second possibility: $C > 0$.
Define, for any natural n, the set

$$A_n = \{\alpha \in \mathbb{R}_+ | h_n(\alpha) > C\} = \{\alpha \in \mathbb{R}_+ | m(\lambda_n)(F_\alpha) > C\}.$$

According to the same Theorem 9.5 from [7], a sufficient condition for the validity of (17) consists in the existence of a natural n_0 such that

$$L(A_{n_0}), \infty. \tag{18}$$

We shall prove that this condition is fulfilled. More precisely, we shall prove a stronger property, namely we shall prove that $L(A_n) < \infty$ for any natural n. Indeed, assuming the existence of some $n \in \mathbb{N}$ such that

$$L(A_n) = \infty \tag{19}$$

we shall arrive at a contradiction, as follows.

For any $0 < \alpha \in A_n$, one has $0 < \beta < \alpha \Rightarrow \beta \in A_n$ (because $F_\beta \supset F_\alpha$). Hence, A_n is an interval with left extremity 0. Condition (19) becomes now: $A_n = \mathbb{R}_+$.

Consequently, for any $\alpha \in \mathbb{R}_+$, one has $m(\lambda_n)(F_\alpha) > C$. Hence, taking a strictly increasing sequence $(\alpha_p)_p$ with $\alpha_p \underset{p}{\longrightarrow} \infty$, one has for any p:

$$m(\lambda_n)(F_{\alpha_p}) > C.$$

Because μ is continuous (being finite) it follows that all measures $m(\lambda_n)$ are finite and continuous, hence

$$\lim_p m(\lambda_n)(F_{\alpha_p}) = m(\lambda_n)(\bigcap_{p=1}^{\infty} F_{\alpha_p}) \geq C.$$

But

$$\bigcap_{p=1}^{\infty} F_{\alpha_p} = \{t \in T | f(t) = \infty\} = \emptyset$$

and we got a contradiction.

□

Comment. Actually, the result obtained in Theorem 3 is a result concerning the commutativity of iterated limits. More precisely:

a) Let $f : T \to \mathbb{R}_+$ be measurable and let $(f_n)_n$ be a sequence of positive simple function which is increasing and has pointwise limit f.

b) It is possible (but not easy) to prove directly that, for any n, one has

$$\lim_{\lambda\to\lambda_0} (S)\int f_n dm(\lambda) = (S)\int f_n dm(\lambda_0). \qquad (\alpha)$$

At the same time, using again Theorem 9.5 from [7], one has

$$\lim_n (S)\int f_n dm(\lambda_0) = (S)\int f dm(\lambda_0). \qquad (\beta)$$

c) Using (α) and (β) we obtain

$$(S)\int f dm(\lambda_0) = \lim_n (S)\int f_n dm(\lambda_0) = \lim_n (\lim_{\lambda\to\lambda_0} (S)\int f_n dm)\lambda).$$

d) On the other hand, the existence of the following iterated limit

$$\lim_{\lambda\to\lambda_0} (\lim_n (S)\int f_n dm(\lambda))$$

and its equality to the iterated limit of previuous point c) would provide Theorem 3.

This kind of proof works in case one of the limits (α), (β) is uniform with respect to the parameter involved in the other limit.

3.2 ***Asymptotic Behaviour***

In this subparagraph, we shall be concerned with the study of the *asymptotic (marginal) measures* obtained as pointwise limits of $m(\lambda)$ when λ tends to -1 or to ∞.

These measures are defined as follows (we shall see that the limits exist):

$$m(-1) : \mathcal{T} \to \mathbb{R}_+,\ m(-1)(A) = \lim_{\lambda\to -1} m(\lambda)(A),$$

$$m(\infty) : \mathcal{T} \to \mathbb{R}_+,\ m(\infty)(A) = \lim_{\lambda\to\infty} m(\lambda)(A).$$

We shall see that the parametric continuity of the integrals continues to hold asymptotically.

The following theorem gathers the most important features of the asymptotic measures.

Theorem 4. *The asymptotic (marginal) measures are computed as follows (for any $A \in \mathcal{T}$):*

$$m(-1)(A) = \begin{cases} 0, & \text{if } \mu(A) = 0 \\ 1, & \text{if } \mu(A) > 0 \end{cases},$$

$$m(\infty)(A) = \begin{cases} 0, & \text{if } \mu(A) < 1 \\ 1, & \text{if } \mu(A) = 1 \end{cases},$$

being both monotone and atomic measures.

For any $-1 < \lambda_1 < \lambda_2 < \infty$, one has $m(-1) \geq m(\lambda_1) \geq m(\lambda_2) \geq m(\infty)$.

Also: $m(-1)$ is countably subadditive and $m(\infty)$ is countably superadditive.

Supplementarily: 1) *In case μ is atomic, one has*

$$m(-1) = m(\infty) = m(\lambda) = \mu$$

for any $\lambda \in (-1, \infty)$. 2) *In case μ is not atomic*: a) *$m(-1)$ is -1-additive (i.e. $m(-1)(A \bigcup B) = m(-1)(A) + m(-1)(B) - m(-1)(A)m(-1)(B)$, whenever A, B are in $\mathcal{T}$ and $A \bigcap B = \emptyset$) and $m(-1)$ is not λ-additive for any $\lambda \in (-1, \infty)$,* b) *$m(\infty)$ is not λ-additive, for any $\lambda \in [-1, \infty)$.*

Proof. Clearly

$$m(-1)(\emptyset) = m(\infty)(\emptyset) = 0$$

and

$$m(-1)(T) = m(\infty)(T) = 1.$$

Now, take $\emptyset \neq A \neq T$, $A \in \mathcal{T}$ (if such A exists).

If $\mu(A) = 0$, one has $m(\lambda)(A) = 0$ for any $\lambda \in (-1, \infty)$ and $m(-1)(A) = 0$. If $\mu(A) > 0$, $\lim\limits_{\lambda \to -1} (\lambda + 1)^{\mu(A)} = 0$, hence $m(-1)(A) = \frac{-1}{-1} = 1$.

If $\mu(A) < 1$, there are two possibilities:

- either $\mu(A) = 0$ and $m(\infty)(A) = 0$;
- or $\mu(A) > 0$, hence $\lim\limits_{\lambda \to \infty} (\lambda + 1)^{\mu(A)} = \infty$ and (L'Hospital)

$$m(\infty)(A) = \lim_{\lambda \to \infty} \mu(A)(\lambda + 1)^{\mu(A) - 1} = 0.$$

If $\mu(A) = 1$, one has $m(\lambda)(A) = 1$ for any $\lambda \in (-1, \infty)$, hence $m(\infty)(A) = 1$.

Now, it is clear that $m(-1)$ and $m(\infty)$ are monotone measures and T is an atom for both (i.e. for any $A \in \mathcal{T}$, the measure of A is either zero or equal to the measure of T).

The fact that, for any $A \in \mathcal{T}$ and any $-1 < \lambda_1 < \lambda_2 < \infty$ one has

$$m(-1)(A) \geq m(\lambda_1)(A) \geq \lambda_2)(A) \geq m(\infty)(A)$$

follows from Lemma 1.

Now let us take a disjoint sequence $(A_n)_n \subset \mathcal{T}$ with $A = \bigcup\limits_{n=1}^{\infty} A_n$.

If $\mu(A_n) = 0$ for any n (hence $m(-1)(A_n) = 0$ for any n and $\mu(A) = 0$, $m(-1)(A) = 0$), we have $m(-1)(A) = 0 = \sum\limits_{n=1}^{\infty} m(-1)(A_n)$. If $\{n|\mu(A_n) > 0\}$ has exactly one element, we have $\mu(A) > 0$, hemce

$$m(-1)(A) = 1 = \sum_{n=1}^{\infty} m(-1)(A_n).$$

If $\{n|\mu(A_n) > 0\}$ has at least two elements, we have

$$m(-1)(A) = 1 < 2 \le \sum_{n=1}^{\infty} m(-1)(A_n).$$

If $\mu(A_n) < 1$ for any n, one has $0 = \sum\limits_{n} m(\infty)(A_n) \le m(\infty)(A)$. If there exists m such that $\mu(A_m) = 1$, we have $\mu(A_n) = 0$ for $n \neq m$ and $m(\infty)(A) = 1 = \sum\limits_{n=1}^{\infty} m(\infty)(A_n)$.

Now, let us assume that μ is atomic. Then clearly

$$m(-1) = m(\infty) = m(\lambda) = \mu$$

for any $\lambda \in (-1, \infty)$. If μ is not atomic, we can find $A, B \in \mathcal{T}$ such that $A \bigcup B = T$, $A \bigcap B = \emptyset$ and $0 < \mu(A) < 1$, $0 < \mu(B) < 1$.

If M, N are in $\mathcal{T}$, $M \bigcap N = \emptyset$, one can analyse all the possibilities and one can see that

$$m(-1)(M \bigcup N) = m(-1)(M) + m(-1)(N) - m(-1)(M)m(-1)(N).$$

On the other hand, if $\lambda \in (-1, \infty)$, it is seen that

$$m(-1)(A \bigcup B) = 1 \neq m(-1)(A) + m(-1)(B) + \lambda m(-1)(A)m(-1)(B) = 2 + \lambda.$$

Also, if $\lambda \in [-1, \infty)$, one has

$$m(\infty)(A \bigcup B) = 1 \neq m(\infty)(A) + m(\infty)(B) + \lambda m(\infty)(A)m(\infty)(B) = 0.$$

□

In the remainder of this subparagraph, we shall consider a strictly monotone sequence $(\lambda_n)_n$. Namely, we shall have: either $(\lambda_n)_n \subset (-1, 0)$, $(\lambda_n)_n$ strictly decreasing, $\lambda_n \xrightarrow[n]{} -1$, or $(\lambda_n)_n \subset (0, \infty)$, $(\lambda_n)_n$ strictly increasing, $\lambda_n \xrightarrow[n]{} \infty$. Using $(\lambda_n)_n$, we define for any n the function $h_n : \mathbb{R}_+ \to \mathbb{R}_+$, via $h_n(\alpha) = m(\lambda_n)(F_\alpha)$. We define also $h_{-1} : \mathbb{R}_+ \to \mathbb{R}_+$, via $h_{-1}(\alpha) = m(-1)(F_\alpha)$ and $h_\infty : \mathbb{R}_+ \to \mathbb{R}_+$, via $h_\infty(\alpha) = m(\infty)(F_\alpha)$. Clearly $0 \le h_n(\alpha) \le 1$ for any n and any α. In case $\lambda_n \underset{n}{\longrightarrow} -1$, the function sequence $(h_n)_n$ is increasing (Lemma 1) with pointwise limit h_{-1}, i.e. $h_{-1}(\alpha) = \lim_n h_n(\alpha) = \sup_n h_n(\alpha)$, for any $\alpha \in \mathbb{R}_+$. In case $\lambda_n \underset{n}{\longrightarrow} \infty$, the function sequence $(h_n)_n$ is decreasing (again Lemma 1) with pointwise limit h_∞, i.e. $h_\infty(\alpha) = \lim_n h_n(\alpha) = \inf_n h_n(\alpha)$, for any $\alpha \in \mathbb{R}_+$.

Concerning the Choquet integral, we have the following *asymptotic continuity* result:

Theorem 5. *Assume $f : T \to \mathbb{R}_+$ is μ-integrable. Then, for any $A \in \mathcal{T}$, one has*

$$\lim_{\lambda \to -1} (C) \int_A f dm(\lambda) = (C) \int_A f dm(-1) \le \infty,$$

$$\lim_{\lambda \to \infty} (C) \int_A f dm(\lambda) = (C) \int_A f dm(\infty) < \infty.$$

Proof. Again, one can work only for $A = T$. As we have seen, for any n the function f is Choquet integrable with respect to $m(\lambda_n)$ and one has

$$(C) \int f dm(\lambda_n) = \int_0^\infty h_n(\alpha) d\alpha.$$

One must prove that, in case $\lambda_n \underset{n}{\longrightarrow} -1$, one has

$$\lim_n \int_0^\infty h_n(\alpha) d\alpha = \int_0^\infty h_{-1}(\alpha) d\alpha = \int f dm(-1) \le \infty$$

and in case $\lambda_n \underset{n}{\longrightarrow} \infty$, one has

$$\lim_n \int_0^\infty h_n(\alpha) d\alpha = \int_0^\infty h_\infty(\alpha) d\alpha = \int f dm(\infty) < \infty.$$

In case $\lambda_n \underset{n}{\longrightarrow} -1$, we use Beppo Levi′s Theorem to obtain that

$$\lim_n \int_0^\infty h_n(\alpha) d\alpha = \sup_n \int_0^\infty h_n(\alpha) d\alpha = \int_0^\infty h_{-1}(\alpha) d\alpha \le \infty$$

(it is possible to have $\int_0^\infty h_{-1}(\alpha) d\alpha = \infty$, as we shall see).

In case $\lambda_n \underset{n}{\longrightarrow} \infty$, we use Lebesgue′s Dominated Convergence Theorem to obtain that

$$\lim \int_0^\infty h_n(\alpha) d\alpha = \inf_n \int_0^\infty h_n(\alpha) d\alpha = \int_0^\infty h_\infty(\alpha) d\alpha < \infty.$$

□

Concerning the Sugeno integral, we have the following *asymptotic continuity* result.

Theorem 6. *Assume $f : T \to \mathbb{R}_+$ is measurable. For any $A \in \mathcal{T}$, one has*

$$\lim_{\lambda \to -1} (S) \int_A f dm(\lambda) = (S) \int_A f dm(-1),$$

$$\lim_{\lambda \to \infty} (S) \int_A f dm(\lambda) = (S) \int_A f dm(\infty).$$

Proof. Again we can work only for $A = T$. We have

$$(S) \int f dm(\lambda_n) = \sup_{\alpha \in [0,1]} g_n(\alpha) \text{ (for any } n)$$

$$(S) \int f dm(-1) = \sup_{\alpha \in [0,1]} g_{-1}(\alpha)$$

$$(S) \int f dm(\infty) = \sup_{\alpha \in [0,1]} g_\infty(\alpha)$$

where $g_n : \mathbb{R}_+ \to \mathbb{R}_+$, $g_n(\alpha) = \alpha \wedge h_n(\alpha)$, $g_{-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $g_{-1}(\alpha) = \alpha \wedge h_{-1}(\alpha)$ and $g_\infty : \mathbb{R}_+ \to \mathbb{R}_+$, $g_\infty(\alpha) = \alpha \wedge h_\infty(\alpha)$.

In case $\lambda_n \underset{n}{\longrightarrow} -1$, the function sequence $(g_n)_n$ is increasing (because $(h_n)_n$ is increasing) and in case $\lambda_n \underset{n}{\longrightarrow} \infty$, the function sequence $(g_n)_n$ is decreasing (because $(h_n)_n$ is decreasing).

One can see that, in case $\lambda_n \underset{n}{\longrightarrow} -1$, one has $g_n \underset{n}{\longrightarrow} g_{-1}$ pointwise and in case $\lambda_n \underset{n}{\longrightarrow} \infty$, $g_n \underset{n}{\longrightarrow} g_\infty$ pointwise. This is true , because for any n and any α, one has

$$g_n(\alpha) = \alpha \wedge h_n(\alpha) = \frac{1}{2}(\alpha + h_n(\alpha) - |\alpha - h_n(\alpha)|)$$

and $h_n(\alpha) \underset{n}{\longrightarrow} h_{-1}(\alpha)$ (in case $\lambda_n \underset{n}{\longrightarrow} -1$) or $h_n(\alpha) \underset{m}{\longrightarrow} h_\infty(\alpha)$ (in case $\lambda_n \underset{n}{\longrightarrow} \infty$).

Consequently, for any $\alpha \in \mathbb{R}_+$ one has

- in case $\lambda_n \underset{n}{\longrightarrow} -1$

$$g_{-1}(\alpha) = \lim_n g_n(\alpha) = \sup_n g_n(\alpha);$$

- in case $\lambda \underset{n}{\longrightarrow} \infty$

$$g_\infty = \lim_n g_n(\alpha) = \inf_n g_n(\alpha).$$

To prove the enunciation, means to prove the following facts:

- in case $\lambda_n \underset{n}{\longrightarrow} -1$

$$\sup_{\alpha\in[0,1]} g_{-1}(\alpha) = \lim_n \sup_{\alpha\in[0,1]} g_n(\alpha)$$

$$\Longleftrightarrow$$

$$\sup_{\alpha\in[0,1]} \sup_n g_n(\alpha) = \sup_n \sup_{\alpha\in[0,1]} g_n(\alpha).$$

and this is true, both members of the stipulated equality being equal to $\sup\{g_n(\alpha)|(\alpha,n) \in [0,1]\times\mathbb{N}\}$.

- in case $\lambda_n \underset{n}{\longrightarrow} \infty$

$$\sup_{\alpha\in[0,1]} g_{\infty}(\alpha) = \lim_n \sup_{\alpha\in[0,1]} g_n(\alpha) \tag{*}$$

$$\Longleftrightarrow$$

$$\sup_{\alpha\in[0,1]} \inf_n g_n(\alpha) = \inf_n \sup_{\alpha\in[0,1]} g_n(\alpha).$$

Unfortunately, only the inequality $\leq$ is generally true and we shall be obliged to exhibit a separate proof of (*), as follows.

We must prove that

$$(S)\int f dm(\infty) = \lim_n (S)\int f dm(\lambda_n). \tag{20}$$

According to the Transformation Theorem (Theorem 9.13, pag. 202 from [7]), we have $(S)\int f dm(\infty) = (S)\int h_\infty dL$ and $(S)\int f dm(\lambda_n) = (S)\int h_n dL$ for any n, hence (20) becomes

$$(S)\int h_\infty dL = \lim_n (S)\int h_n dL \tag{20'}$$

The proof of (20′) is exactly the proof of (17) for the case of $(\lambda_n)_n$ strictly increasing, replacing: a) λ_0 with ∞ (hence $(\lambda_n)_n$ is a strictly increasing sequence such that $\lambda_n \underset{n}{\longrightarrow} \infty$); b) h with h_∞ (hence $(h_n)_n$ is a decreasing sequence with pointwise limit h_∞).

□

Conclusion. The results obtained up to now can be unitarily expressed in the following

Synthesis Theorem. *Let $f : T \to \mathbb{R}_+$ be a measurable function (which is assumed to be μ-integrable in case of the Choquet integral). Let also $\lambda_0 \in [-1,\infty]$. One has, for any $A \in \mathcal{T}$:*

$$(C)\int_A f dm(\lambda_0) = \lim_{\lambda\to\lambda_0} (C)\int_A f dm(\lambda),$$

$$(S)\int_A fdm(\lambda_0) = \lim_{\lambda\to\lambda_0}(S)\int_A fdm(\lambda).$$

3.3 *Some Computations*

Throughout this subparagraph we shall work within the following framework:

- The measure space (probabilistic space) $(T, \mathcal{T}, \mu)$ is given as follows: $T = [1,\infty)$, $\mathcal{T} =$ the Lebesgue measurable sets of $[1,\infty)$ and $\mu : \mathcal{T} \to \mathbb{R}_+$ is the probability given via

$$\mu(A) = \int_A \varphi dL$$

(here L is the Lebesgue measure on $[1,\infty)$ and $\varphi : [1,\infty) \to \mathbb{R}_+$ acts via $\varphi(t) = \frac{1}{t^2}$).

- Let us fix a real number p. We shall work with the measurable function $f : [1,\infty) \to \mathbb{R}_+$ acting via $f(t) = t^p$. It is seen that f is μ-integrable if and only if $p < 1$.

a) We begin with the computation, for a fixed $\alpha \in \mathbb{R}_+$, of the set

$$F_\alpha = \{t \in [1,\infty) | f(t) = t^p \geq \alpha\}.$$

We have: For $p > 0$, $F_\alpha = \begin{cases} [1,\infty), & \text{if } 0 \leq \alpha \leq 1 \\ [\alpha^{1/p},\infty), & \text{if } \alpha > 1 \end{cases}$.

For $p = 0$, $F_\alpha = \begin{cases} [1,\infty), & \text{if } 0 \leq \alpha \leq 1 \\ \emptyset, & \text{if } \alpha > 1 \end{cases}$.

For $p < 0$, $F_\alpha = \begin{cases} [1,\infty), & \text{if } \alpha = 0 \\ [1,\alpha^{1/p}], & \text{if } 0 < \alpha \leq 1 \\ \emptyset, & \text{if } \alpha > 1 \end{cases}$.

b) For any $1 \leq a < b < \infty$ and any $0 \neq \lambda \in (-1,\infty)$:

$$\mu([a,b]) = \frac{1}{a} - \frac{1}{b} \Rightarrow m(\lambda)([a,b]) = \frac{(\lambda+1)^{1/a-1/b} - 1}{\lambda},$$

$$\mu([a,\infty)) = \frac{1}{a} \Rightarrow m(\lambda)([a,\infty)) = \frac{(\lambda+1)^{1/a} - 1}{\lambda}.$$

Consequently, for any $\alpha \in \mathbb{R}_+$ and any $\lambda \in (-1,\infty)$, one can compute the value of $m(\lambda)(F_\alpha)$:

- For $\lambda = 0$, one has $m(\lambda) = m(0) = \mu$, hence $m(0)(F_\alpha) = \mu(F_\alpha)$:
For $p > 0$, $m(0)(F_\alpha) = \mu(F_\alpha) = \begin{cases} 1, & \text{if } 0 \leq \alpha \leq 1 \\ \alpha^{-1/p}, & \text{if } \alpha > 1 \end{cases}$.
For $p = 0$, $m(0)(F_\alpha) = \mu(F_\alpha) = \begin{cases} 1, & \text{if } 0 \leq \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases}$.
For $p < 0$, $m(0)(F_\alpha) = \mu(F_\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ 1 - \alpha^{-1/p}, & \text{if } 0 < \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases}$.
- For $\lambda \neq 0$, one has:
For $p > 0$, $m(\lambda)(F_\alpha) = \begin{cases} 1, & \text{if } 0 \leq \alpha \leq 1 \\ \frac{(\lambda+1)^{\alpha^{-1/p}}-1}{\lambda}, & \text{if } \alpha > 1 \end{cases}$.
For $p = 0$, $m(\lambda)(F_\alpha) = \begin{cases} 1, & \text{if } 0 \leq \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases}$.
For $p < 0$, $m(\lambda)(F_\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ \frac{(\lambda+1)^{1-\alpha^{-1/p}}-1}{\lambda}, & \text{if } 0 < \alpha \leq 1 \\ 0, & \text{if } \alpha > 1 \end{cases}$.

c) Hence, one can (theoretically) compute (C) $\int f dm(\lambda)$ and $(S) \int f dm(\lambda)$ for any $\lambda \in (-1, \infty)$. We exhibit these computations:

- For $p > 0$:

$$(C) \int f dm(0) = (C) \int f d\mu = \int f d\mu = \int_0^\infty t^p \frac{1}{t^2} dt = \frac{1}{1-p}$$

(in this case we work for $p < 1$)

$$(S) \int f dm(0) = \sup_{\alpha \in \mathbb{R}_+} (\alpha \wedge \mu(F_\alpha)) = \sup_{\alpha \in [0,1]} (\alpha \wedge \mu(F_\alpha) = 1$$

(Indeed: For $0 \leq \alpha \leq 1$, $\mu(F_\alpha) = 1$ and $\alpha \wedge \mu(F_\alpha) = \alpha$)
and, for $\lambda \neq 0$:

$$(C) \int f dm(\lambda) = \int_0^\infty m(\lambda)(F_\alpha) d\alpha =$$
$$= \int_0^1 m(\lambda)(F_\alpha) d\alpha + \int_1^\infty m(\lambda)(F_\alpha) d\alpha = 1 + \int_1^\infty \frac{(\lambda+1)^{\alpha^{-1/p}}-1}{\lambda} d\alpha,$$

(in this case we work for $p < 1$).

$$(S) \int f dm(\lambda) = 1.$$

(Indeed: For $0 \leq \alpha \leq 1$, $m(\lambda)(F_\alpha) = 1$ and $\alpha \wedge m(\lambda)(F_\alpha) = \alpha$.)

- For $p = 0$:
We have $f \equiv 1$.

$$(C)\int fdm(0) = (C)\int 1d\mu = \int 1d\mu = 1$$

$$(S)\int fdm(0) = (S)\int 1d\mu = \sup_{\alpha\in\mathbb{R}_+}(\alpha\wedge\mu(F_\alpha)) = \sup_{\alpha\in[0,1]}\alpha = 1$$

and, for $\lambda\neq 0$:

$$(C)\int fdm(\lambda) = \int_0^1 1d\alpha = 1$$

$$(S)\int fdm(\lambda) = \sup_{\alpha\in\mathbb{R}_+}(\alpha\wedge m(\lambda)(F_\alpha)) = \sup_{\alpha\in[0,1]}\alpha = 1$$

- For $p<0$:

$$(C)\int fdm(0) = (C)\int fd\mu = \int fd\mu = \int_1^\infty t^p\frac{1}{t^2}dt = \frac{1}{1-p}$$

$$(S)\int fdm(0) = \sup_{\alpha\in\mathbb{R}_+}(\alpha\wedge\mu(F_\alpha)) = \alpha_0.$$

Here: α_0 is the unique solution (in $[0,1]$) of the equation $\alpha = 1-\alpha^{-1/p}$. For instance, if $p=-1$, we have $\alpha_0 = \frac{1}{2} = (S)\int fdm(0)$.
(Indeed, we want to find $\sup_{\alpha\in[0,1]}\alpha\wedge(1-\alpha^{-1/p})$. The continuous function $\alpha\longmapsto\alpha-(1-\alpha^{-1/p}) = \alpha+\alpha^{-1/p}-1$ is strictly increasing from -1 to 1 on $[0,1]$ and has an unique zero equal to $\alpha_0\in(0,1)$. On $[0,\alpha_0]$ one has $\alpha\wedge(1-\alpha^{-1/p}) = \alpha$ and on $[\alpha_0,1]$ one has $\alpha\wedge(1-\alpha^{-1/p}) = 1-\alpha^{-1/p}$.)
and, for $\lambda\neq 0$:

$$\begin{aligned}(C)\textstyle\int fdm(\lambda) &= \textstyle\int_0^\infty m(\lambda)(F_\alpha)d\alpha = \int_0^1 m(\lambda)(F_\alpha)d\alpha = \\ &= \textstyle\int_0^1 \frac{(\lambda+1)^{1-\alpha^{-1/p}}-1}{\lambda}d\alpha.\end{aligned}$$

$$(S)\int fdm(\lambda) = \sup_{\alpha\in\mathbb{R}_+}(\alpha\wedge m(\lambda)(F_\alpha)) = \sup_{\alpha\in[0,1]}(\alpha\wedge m(\lambda)(F_\alpha)) = \alpha_0.$$

Here: α_0 is the unique solution (in $[0,1]$) of the equation

$$\alpha = \frac{(\lambda+1)^{1-\alpha^{-1/p}}-1}{\lambda}.$$

(Indeed, the continuous function $\alpha\longmapsto\frac{(\lambda+1)^{1-\alpha^{-1/p}}-1}{\lambda}$ is strictly decreasing on $[0,1]$, hence the function $\alpha\longmapsto\alpha-\frac{(\lambda+1)^{1-\alpha^{-1/p}}-1}{\lambda}$ is strictly increasing from -1 to 1 on $[0,1]$ and has an unique zero $\alpha_0\in(0,1)$ ...)

d) Let us turn to the marginal measures. Fix arbitrarily $\alpha \in \mathbb{R}_+$.

- For $p > 0$:

$$h_{-1}(\alpha) = m(-1)(F_\alpha) = 1,$$

$$h_\infty(\alpha) = m(\infty)(F_\alpha) = \begin{cases} 1, & \text{if } 0 \le \alpha \le 1 \\ 0, & \text{if } \alpha > 1 \end{cases}.$$

- For $p = 0$:

$$h_{-1}(\alpha) = m(-1)(F_\alpha) = \begin{cases} 1, & \text{if } 0 \le \alpha \le 1 \\ 0, & \text{if } \alpha > 1 \end{cases}.$$

$$h_\infty(\alpha) = m(\infty)(F_\alpha) = \begin{cases} 1, & \text{if } 0 \le \alpha \le 1 \\ 0, & \text{if } \alpha > 1 \end{cases}.$$

- For $p < 0$:

$$h_{-1}(\alpha) = m(-1)(F_\alpha) = \begin{cases} 1, & \text{if } 0 \le \alpha \le 1 \\ 0, & \text{if } \alpha > 1 \end{cases}.$$

$$h_\infty(\alpha) = m(\infty)(F_\alpha) = \begin{cases} 1, & \text{if } \alpha = 0 \\ 0, & \text{if } \alpha > 0 \end{cases}.$$

e) We can compute the Choquet and Sugeno integrals with respect to the marginal measures.

$$(C)\int f dm(-1) = \int_0^\infty h_{-1}(\alpha) d\alpha = \begin{cases} \infty, & \text{if } p > 0 \\ 1, & \text{if } p \le 0 \end{cases}$$

(for $p > 0$ we obtain the value ∞, even in the case $0 < p < 1$).

$$(C)\int f dm(\infty) = \int_0^\infty h_\infty(\alpha) d\alpha = \begin{cases} 1, & \text{if } p \ge 0 \\ 0, & \text{if } p < 0 \end{cases}$$

To compute the Sugeno integrals (all of them are equal to 1, with the exception of $(S)\int f dm(\infty) = 0$):

- For $p > 0$:

$$h_{-1}(\alpha) \wedge \alpha = \alpha, \text{ if } 0 \le \alpha \le 1$$

hence

$$(S)\int f dm(-1) = \sup_{\alpha \in [0.1]} (h_{-1}(\alpha) \wedge \alpha) = 1.$$

$$h_\infty(\alpha) \wedge \alpha = \alpha, \text{ if } 0 \le \alpha \le 1$$

hence

$$(S)\int f dm(\infty) = \sup_{\alpha \in [0.1]} (h_\infty(\alpha) \wedge \alpha) = 1.$$

- For $p = 0$:

$$h_{-1}(\alpha) \wedge \alpha = \alpha, \text{ if } 0 \leq \alpha \leq 1$$

hence

$$(S) \int f dm(-1) = \sup_{\alpha \in [0.1]} (h_{-1}(\alpha) \wedge \alpha) = 1.$$

$$h_{\infty}(\alpha) \wedge \alpha = \alpha, \text{ if } 0 \leq \alpha \leq 1$$

hence

$$(S) \int f dm(\infty) = \sup_{\alpha \in [0.1]} (h_{\infty}(\alpha) \wedge \alpha) = 1.$$

- For $p < 0$:

$$h_{-1}(\alpha) \wedge \alpha = \alpha, \text{ if } 0 \leq \alpha \leq 1$$

hence

$$(S) \int f dm(-1) = \sup_{\alpha \in [0.1]} (h_{-1}(\alpha) \wedge \alpha) = 1.$$

$$h_{\infty}(\alpha) \wedge \alpha = 0, \text{ for any } \alpha \in [0, \infty)$$

hence

$$(S) \int f dm(\infty) = 0.$$

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