# A Characterization of Einstein Spaces

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**Abstract**
*The aim of this article is to establish a property of Einstein spaces in terms of scalar curvature of $k$-planes included in the tangent space where $k$ is less than the dimension of the space.*

**Keywords:** *scalar curvature, Einstein space.*

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The aim of this paper is to establish some properties of Einstein spaces in terms of scalar curvatures of $k$-planes included in the tangent space. A characterizations of the 4-dimensional Einstein spaces is given in ([3]) and generalizations of that to any dimension are given in ([1]) and ([2]). In this paper we generalize the main theorem from ([2]).

Let $(M, g_M)$ be a Riemannian $m$-manifold and $p \in M$. For $L \subset T_pM$ a subspace of the tangent space of dimension $r \leq m$, which is also called an $r$-plane, we will consider $\{e_1, ..., e_r\}$ an orthonormal basis for $L$ and by $\tau(L) = \sum\limits_{1 \leq i < j \leq r} K(e_i \wedge e_j)$ we will denote the scalar curvature of $L$, where $K(e_i \wedge e_j)$ is the sectional curvature of the plane spanned by $\{e_i, e_j\}$ and by $\tau(L^\perp) = \sum\limits_{r+1 \leq i < j \leq m} K(e_i \wedge e_j)$ the scalar curvature of $L^\perp$, where $L^\perp$ is the orthogonal complement of $L$. We have the following well-known definition.

**Definition 1.** A Riemannian manifold $(M, g_M)$ is said to be *Einstein* if its Ricci tensor is proportional to the metric, that is $Ric_M = \lambda g_M$.

We consider now that $\dim(M) = m = 2n + 1$, $n > 1$ and we have the following:

**Theorem 2.** *Let $(M, g_M)$ be a Riemannian $(2n + 1)$-manifold, $n > 1$. Then $(M, g_M)$ is an Einstein space satisfying $Ric_M = \lambda g_M$ if and only if $\tau(L^\perp) - \tau(L) = \left(n - k + \frac{1}{2}\right)\lambda$ for every $k$-plane $L \subset T_pM$ and every $k \in \{2, 3, ..., n\}$, where $L^\perp$ denotes the orthogonal complement of $L$ and $p \in M$.*

*Proof:* " $\Longrightarrow$ " Let $p \in M$ and $L \subset T_pM$ be a subspace of dimension $k \in \{2, .., n\}$. Let $\mathcal{B} = \{e_1, ..., e_k\}$ be an orthonormal basis of $L$. We complete $\mathcal{B}$ to $\mathcal{B}^{'} = \{e_1, .., e_k, e_{k+1} .., e_{2n+1}\}$ an orthonormal basis of $T_pM$. Then $\mathcal{B}^{''} = \{e_{k+1}, ..., e_{2n+1}\}$ will be an orthonormal basis of $L^\perp$. We know that

$Ric_M(X,Y) = \sum_{i=1}^{2n+1} R(X,e_i,Y,e_i) = \lambda g_M(X,Y)$ for every $X,Y \in \Gamma(TM)$. Thus for $X=Y=e_i$ we have that

$$Ric_M(e_i) = Ric_M(e_i,e_i) = \sum_{j=1}^{2n+1} R(e_i,e_j,e_i,e_j) = \lambda$$

for every $i \in \{1,...,2n+1\}$.

We will write now all the equations and by the formula of the Ricci curvature we will have the following system:

$$\begin{cases} (1) & Ric_M(e_1) = K(e_1\wedge e_2) + K(e_1\wedge e_3) + \ldots + K(e_1\wedge e_{2n+1}) = \lambda \\ (2) & Ric_M(e_2) = K(e_2\wedge e_1) + K(e_2\wedge e_3) + \ldots + K(e_2\wedge e_{2n+1}) = \lambda \\ (3) & Ric_M(e_3) = K(e_3\wedge e_1) + K(e_3\wedge e_2) + \ldots + K(e_3\wedge e_{2n+1}) = \lambda \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (2n) & Ric_M(e_{2n}) = K(e_{2n}\wedge e_1) + K(e_{2n}\wedge e_2) + \ldots + (e_{2n}\wedge e_{2n+1}) = \lambda \\ (2n+1) & Ric_M(e_{2n+1}) = K(e_{2n+1}\wedge e_1) + K(e_{2n+1}\wedge e_2) + \ldots + \\ & \qquad +K(e_{2n+1}\wedge e_{2n}) = \lambda \end{cases}$$

By summing the first $k$ equations, we obtain the relation:

$$2\tau(L) + \sum_{1\le i\le k<j\le 2n+1} K(e_i\wedge e_j) = k\lambda \qquad (1)$$

By summing the last $2n+1-k$ equations, we obtain the relation:

$$2\tau(L^\perp) + \sum_{1\le j\le 2n+1-k<i\le 2n+1} K(e_i\wedge e_j) = (2n+1-k)\lambda \qquad (2)$$

Now, making the difference $(2)-(1)$ and remarking that

$$\sum_{1\le i\le k<j\le 2n+1} K(e_i\wedge e_j) = \sum_{1\le j\le 2n+1-k<i\le 2n+1} K(e_i\wedge e_j)$$

we obtain the following:

$$(2n+1-2k)\lambda =$$
$$= 2\tau(L^\perp) - 2\tau(L) + \sum_{1\le j\le k<i\le 2n+1} K(e_i\wedge e_j) - \sum_{1\le i\le 2n+1-k<j\le 2n+1} K(e_i\wedge e_j) =$$
$$= 2\tau(L^\perp) - 2\tau(L)$$

This implies that $\tau(L^\perp) - \tau(L) = \left(n-k+\frac{1}{2}\right)\lambda$ for every $k$-plane $L \subset T_pM$ and every $k \in \{2,3,...,n\}$.

"$\Longleftarrow$" Let $k=n$, $p\in M$ and $\mathcal{B} = \{e_1,...,e_n,e_{n+1},...,e_{2n+1}\}$ be an orthonormal basis of $T_pM$. Let $L = Sp(\{e_{n+2},...,e_{2n+1}\})$ and $L_0 = Sp(\{e_2,...,e_{n+1}\})$. Then $L^\perp = Sp(\{e_1,...,e_{n+1}\})$ and $L_0^\perp = Sp(\{e_1,e_{n+2},...,e_{2n+1}\})$. Then

$$Ric_M(e_1) =$$

$$= [K(e_1 \wedge e_2) + \ldots + K(e_1 \wedge e_{n+1})] + [K(e_1 \wedge e_{n+2}) + \ldots + K(e_1 \wedge e_{2n+1})] =$$

$$= \left[\tau(L^{\perp}) - \sum_{2 \le i < j \le n+1} K(e_i \wedge e_j)\right] + \left[\tau(L_0^{\perp}) - \sum_{n+2 \le i < j \le 2n+1} K(e_i \wedge e_j)\right] =$$

$$= \left[\tau(L) + \frac{\lambda}{2} - \tau(L_0)\right] + \left[\tau(L_0) + \frac{\lambda}{2} - \tau(L)\right] = \lambda.$$

In the same way we obtain $Ric_M(e_i) = \lambda$ for every $i = \{2, ..., 2n+1\}$ and then $Ric_M(X,X) = \lambda g_M(X,X)$ for every $X \in \Gamma(TM)$. Because the tensors $Ric_M$ and $g_M$ are symmetric, it follows that $Ric_M(X,Y) = \lambda g_M(X,Y)$ for every $X, Y \in \Gamma(TM)$ and then $M$ is an Einstein space of constant $\lambda$.

A similar result can be obtain for Einstein spaces of even dimension.

**Theorem 3.** *Let $(M, g_M)$ be a Riemannian $(2n)$-manifold, $n > 1$. Then $(M, g_M)$ is an Einstein space satisfying $Ric_M = \lambda g_M$ if and only if $\tau(L^{\perp}) - \tau(L) = (n-k)\,\lambda$ for every $k$-plane $L \subset T_pM$ and every $k \in \{2, 3, ..., n\}$, where $L^{\perp}$ denotes the orthogonal complement of $L$ and $p \in M$.*

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