About Generalized Quasi-Einstein Spaces

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Abstract

The aim of this paper is to give a characterisation of generalized quasi-Einstein manifolds in terms of scalar curvatures of the subspaces of the tangent space.

Keywords: *quasi-Einstein manifold, generalized quasi-Einstein.*

MSC 2000 Classification: 53C15, 53B20, 53C42.

According to ([2]) we have the following definition.

Definition 1. A non-flat Riemannian manifold (M, g), $n > 2$, is said to be a *quasi-Einstein* manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and satisfies the condition $Ric(X, Y) = ag(X, Y) + bA(X)A(Y)$ for every $X, Y \in \Gamma(TM)$, where a, b are real scalars, $b \neq 0$ and A is a non-zero 1-form on M such that $A(X) = g(X, U)$ for all vector field $X \in \Gamma(TM)$, U being an unit vector field which is called the generator of the manifold.

According to ([4]) we have the following definition.

Definition 2. A non-flat Riemannian manifold (M, g), $n > 2$, is said to be a *generalized quasi-Einstein* manifold if its Ricci tensor Ric of type $(0, 2)$ is not identically zero and satisfies the condition

$$Ric_M(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y)$$

for every $X, Y \in \Gamma(TM)$ where a, b, c are real scalars, $b \neq 0, c \neq 0$ and A, B are non-zero 1-form on M such that $A(X) = g(X, U)$, $B(X) = g(X, V)$, $g(U, V) = 0$ for all vector field $X \in \Gamma(TM)$, U, V being unit vector fields which are called the generators of the manifold.

Let M be a Riemannian n-manifold. Let $p \in M$ and $L \subset T_pM$ a subspace of dimension $r \leq n$. Let $\{e_1, ..., e_r\}$ be a basis for L. We will denote by $\tau(L) = \sum\limits_{1 \leq i < j \leq r} K(e_i \wedge e_j)$, where $K(e_i \wedge e_j)$ is the sectional curvature of the plane determinated by $\{e_i, e_j\}$. $\tau(L)$ is called the scalar curvature of L. In these conditions, the orthoganal complement of L is the plane spanned by $\{e_{r+1}, ..., e_n\}$ and is denoted by $L^{\perp}$.

Characterizations of Einstein spaces are given as following: the Einstein spaces are characterized in ([3], [6]), the quasi-Einstein spaces are studied in ([1], [2]) and the generalized quasi-Einstein spaces are studied in ([4], [5]).

The aim of this paper is to extend theorem 2.1 from ([5]). Thus, we will give a characterisation of generalized quasi-Einstein manifolds of dimension $2n+1$, $n \geq 2$ in terms of scalar curvatures of k-planes included in the tangent space for every $k \in \{2, ..., n\}$.

Theorem 3. *Let (M, g) be a Riemannian $(2n+1)$-manifold, $n \geq 2$. Then the following conditions are equivalent:*

1). *M is a generalized quasi-Einstein manifold with:*

$$Ric(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y)$$

for every $X, Y \in \Gamma(TM)$, where a, b, c are real scalars and A, B are non-zero 1-forms on M such that $A(X) = g(X, U)$, $B(X) = g(X, V)$, $g(U, V) = 0$ for all vector field $X \in \Gamma(TM)$ with U, V being unit vector fields.

2). *The scalar curvature of the k-planes included in the tangent space verify:*

$\tau\left(L_1^{\perp}\right) = \tau(L_1) + \left(n - k + \frac{1}{2}\right) a + \frac{1}{2}(b + c)$ *for any k-plane section $L_1 \subset T_pM$ such that $U, V \notin L_1$ and every $k \in \{2, ..., n\}$,*

$\tau(L_2^{\perp}) = \tau(L_2) + \left(n - k + \frac{1}{2}\right) a - \frac{1}{2}(b + c)$ *for any k-plane section $L_2 \subset T_pM$ such that $U, V \in L_2$ and every $k \in \{2, ..., n\}$,*

$\tau(L_3^{\perp}) = \tau(L_3) + \left(n - k + \frac{1}{2}\right) a + \frac{1}{2}(b - c)$ *for any k-plane section $L_3 \subset T_pM$ such that $U \notin L_3, V \in L_3$ and every $k \in \{2, ..., n\}$,*

$\tau(L_4^{\perp}) = \tau(L_4) + \left(n - k + \frac{1}{2}\right) a - \frac{1}{2}(b - c)$ *for any k-plane section $L_4 \subset T_pM$ such that $U \in L_4, V \notin L_4$ and every $k \in \{2, ..., n\}$,*

where $L^{\perp}$ denotes the orthogonal complement of L in T_pM for every $p \in M$.

Proof. 1) $\Longrightarrow$ 2) Let $p \in M$ and $\{e_1, .., e_n, ..., e_{2n+1}\}$ be an orthonormal frame of T_pM such that $U = e_1$ and $V = e_2$. We know that

$$Ric(X, Y) = \sum_{i=1}^{2n+1} R(X, e_i, Y, e_i) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y)$$

for every $X, Y \in \Gamma(TM)$. Let $X = Y = e_i$. This implies that

$$Ric(e_i) = Ric(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_i, e_j, e_i, e_j) = a$$

for every $i \in \{3, ..., 2n+1\}$. In the same way we obtain that

$$Ric(U) = Ric(e_1) = a + b$$

and

$$Ric(V) = Ric(e_2) = a + c.$$

Now, let $k \in \{2, ..., n\}$ be fixed.

We will write all the equations and by the formula of Ricci curvature, we will have the following system of $2n+1$ equations:

$$\begin{cases} 1) & Ric(e_1) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \ldots + K(e_1 \wedge e_{2n+1}) = a + b \\ 2) & Ric(e_2) = K(e_2 \wedge e_1) + K(e_2 \wedge e_3) + \ldots + K(e_2 \wedge e_{2n+1}) = a + c \\ 3) & Ric(e_3) = K(e_3 \wedge e_1) + K(e_3 \wedge e_2) + \ldots + K(e_3 \wedge e_{2n+1}) = a \\ \cdots & \cdots \\ 2n+1) & Ric(e_{2n+1}) = K(e_{2n+1} \wedge e_1) + K(e_{2n+1} \wedge e_2) + \ldots + \\ & \qquad + K(e_{2n+1} \wedge e_{2n}) = a \end{cases}$$

Without any loss of generality we will consider the following k-planes included in the T_pM : $L_1 = Sp(\{e_3, e_4, ..., e_{k+2}\})$, $L_2 = Sp(\{e_1, e_2, ..., e_k\})$, $L_3 = Sp(\{e_2, e_3, ..., e_{k+1}\})$, $L_4 = Sp(\{e_1, e_3, e_4, ..., e_{k+1}\})$.

Now, by summing the first k equations we have the following relation:

$$2\tau(L_2) + \sum_{1 \le i \le k < j \le 2n+1} K(e_i \wedge e_j) = ka + b + c \qquad (i)$$

By summing the last $2n + 1 - k$ equations we have another relation:

$$2\tau(L_2^{\perp}) + \sum_{1 \le j \le k < i \le 2n+1} K(e_i \wedge e_j) = (2n + 1 - k)a \qquad (ii)$$

Then $(ii) - (i)$ implies:

$$(2n + 1 - k)a - ka - b - c =$$

$$2\tau(L_2^{\perp}) - 2\tau(L_2) + \sum_{1 \le j \le k < i \le 2n+1} K(e_i \wedge e_j) - \sum_{1 \le i \le k < j \le 2n+1} K(e_i \wedge e_j) \Longrightarrow$$

$$\tau(L_2^{\perp}) = \tau(L_2) + \left(n - k + \frac{1}{2}\right) a - \frac{1}{2}(b + c).$$

In a similar way, by summing the equations from 3) to $k + 2$) we have:

$$2\tau(L_1) + \sum_{3 \le i \le k+2 < j \le 2n+1} K(e_i \wedge e_j) + \sum_{1 \le j < 3 \le i \le k+2} K(e_i \wedge e_j) = ka \qquad (iii)$$

Also, by summing the first two equations with the equations from $k+3$) to $2n + 1$) we have:

$$2\tau(L_1^{\perp}) + \sum_{3 \le i \le k+2 < j \le 2n+1} K(e_i \wedge e_j) + \sum_{1 \le j < 3 \le i \le k+2} K(e_i \wedge e_j) =$$

$$= (2n + 1 - k)a + b + c \qquad (iv)$$

Then $(iv) - (iii)$ implies:

$$(2n+1-k)a+b+c-ka = 2\tau(L_1^\perp) - 2\tau(L_1) \Longrightarrow$$

$$\tau(L_1^\perp) = \tau(L_1) + \left(n-k+\frac{1}{2}\right)a + \frac{1}{2}(b+c).$$

In a similar way, by summing the equations from 2) to $k+1$) we have:

$$2\tau(L_3) + \sum_{2\le i\le k+1<j\le 2n+1} K(e_i\wedge e_j) + \sum_{2\le i\le k+1} K(e_i\wedge e_1) = ka+c \qquad (v)$$

Also, by summing the first equation with the equations from $k+2$) to $2n+1$) we have:

$$2\tau(L_3^\perp) + \sum_{2\le i\le k+1<j\le 2n+1} K(e_i\wedge e_j) + \sum_{2\le i\le k+1} K(e_i\wedge e_1) = (2n+1-k)a+b \qquad (vi)$$

Then $(vi)-(v)$ implies:

$$(2n+1-k)a+b-ka-c = 2\tau(L_3^\perp) - 2\tau(L_3) \Longrightarrow$$

$$\tau(L_3^\perp) = \tau(L_3) + \left(n-k+\frac{1}{2}\right)a + \frac{1}{2}(b-c).$$

In a similar way, by summing the first equation with all the equations from 3) to $k+1$) we have:

$$2\tau(L_4) + \sum_{1\le i\le k+1<j\le 2n+1} K(e_i\wedge e_j) - \sum_{1\le i\le k+1} K(e_i\wedge e_2) = ka+b \qquad (vii)$$

Also, by summing the second equation with the equations from $k+2$) to $2n+1$) we have:

$$2\tau(L_4^\perp) + \sum_{1\le i\le k+1<j\le 2n+1} K(e_i\wedge e_j) - \sum 1\le i\le k+1 K(e_i\wedge e_2) =$$

$$= (2n+1-k)a+c \qquad (viii)$$

Then $(viii)-(vii)$ implies:

$$(2n+1-k)a+c-ka-b = 2\tau(L_4^\perp) - 2\tau(L_4) \Longrightarrow$$

$$\tau(L_4^\perp) = \tau(L_4) + \left(n-k+\frac{1}{2}\right)a - \frac{1}{2}(b-c).$$

2) $\Longrightarrow$ 1) Let $k=n$ and $p\in M$. Let also $\{e_1,...,e_n,e_{n+1},...,e_{2n+1}\}$ be an orthonormal frame of T_pM such that $U=e_1$ and $V=e_2$. Let $L=Sp(\{e_{n+2},...,e_{2n+1}\})$ and $L_0=Sp(\{e_2,...,e_{n+1}\})$. Then $L^\perp = Sp(\{e_1,...,e_{n+1}\})$ and $L_0^\perp = Sp(\{e_1,e_{n+2},...,e_{2n+1}\})$. Thus:

$$Ric(e_1) =$$

$$= [K(e_1 \wedge e_2) + \ldots + K(e_1 \wedge e_{n+1})] + [K(e_1 \wedge e_{n+2}) + \ldots + K(e_1 \wedge e_{2n+1})] =$$

$$= \left[\tau(L^{\perp}) - \sum_{2 \leq i < j \leq n+1} K(e_i \wedge e_j)\right] + \left[\tau(L_0^{\perp}) - \sum_{n+2 \leq i < j \leq 2n+1} K(e_i \wedge e_j)\right] =$$

$$= \left[\tau(L) + \frac{1}{2}(a+b+c) - \tau(L_0)\right] + \left[\tau(L_0) + \frac{1}{2}(a+b-c) - \tau(L)\right] = a + b.$$

In a similar way one can prove that $Ric(e_2) = a + c$ and $Ric(e_i) = a$ for every $i \in \{3, ..., 2n+1\}$.

We define now the 1-forms $A(X) = g(X, U)$, $B(X) = g(X, V)$ such that $A(U) = B(V) = 1$ and we consider the $(0, 2)$-tensor

$$P(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y).$$

Then $Ric(X, X) = P(X, X)$ for every $X \in \Gamma(TM)$. Because the tensors Ric and P are symmetric, it follows that $Ric(X, Y) = P(X, Y)$ for every $X, Y \in \Gamma(TM)$ and then M is a generalized quasi-Einstein manifold.

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