On the connectedness properties of the attractors of iterated function systems

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Abstract
This paper is a survey and treats some of the topological properties of the attractors of iterated function systems (finite and infinite) such as connectedness, arcwise connectedness, locally arcwise connectedness and other properties of this type.
Keywords: *attractors, iterated function systems, connectednes, arcwise connectedness.*
MSC 2010 Classification: 28A80.

1. INTRODUCTION

We start with a metric space (X, d) and we denote by $\mathcal{K}(X)$ the set of nonempty compact subsets of X, by $\mathcal{B}(X)$ the set of nonempty bounded and closed subsets of X and by $\mathcal{P}(X)$ the set of nonempty subsets of X. For a set $A \subset X$ we denote by $d(A)$ the diameter of A, that is $d(A) = \sup_{x,y\in A} d(x, y)$.

Definition 1.1. *Let (X, d) be a metric space. The application $h : \mathcal{K}(X)\times\mathcal{K}(X) \longrightarrow [0, +\infty)$ defined by $h(A, B) = \max(d(A, B), d(B, A))$, where $d(A, B) = \sup_{x\in A} d(x, B) = \sup_{x\in A} (\inf_{y\in B} d(x, y))$ is called the **Hausdorff-Pompeiu metric**.*

Proposition 1.1. ([1], [5], [19]) *Let (X, d_X) and (Y, d_Y) two metric spaces. Then:*

1) *If H and K are two nonempty subsets of X then $h_X(H, K) = h_X(\overline{H}, \overline{K})$, where h_X is the the Hausdorff-Pompeiu semidistance associated to distance d_X.*

2) *If $(H_i)_{i\in I}$ and $(K_i)_{i\in I}$ are two families of nonempty subsets of X then $h_X(\cup_{i\in I}H_i, \cup_{i\in I}K_i) = h_X\left(\overline{\cup_{i\in I}H_i}, \overline{\cup_{i\in I}K_i}\right) \le \sup_{i\in I} h_X(H_i, K_i)$.*

3) *If H and K are two nonempty subsets of X and $f : X \longrightarrow Y$ a function then $h_Y(f(K), f(H)) \le Lip(f) \cdot h_X(K, H)$.*

4) *If $(H_n)_{n\ge1} \subset \mathcal{P}(X)$ is a sequence of sets of X, and $H \in \mathcal{P}(X)$ is a set such that $h_X(H, H_n) \longrightarrow 0$, then a element $x \in X$ belongs to H if and only if there exists $x_n \in H_n$ for every $n \ge 1$ such that $x_n \longrightarrow x$.*

5) *If $(H_n)_{n\geq 1} \subset \mathcal{P}(X)$ is a sequence of relatively compact sets and $H \in \mathcal{P}(X)$ is a set such that $h_X(H, H_n) \longrightarrow 0$, then H is a relatively compact set.*

6) *If $(H_n)_{n\geq 1} \subset \mathcal{P}(X)$ is a sequence of compact connected sets and $H \in \mathcal{P}(X)$, is a closed set such that $h_X(H, H_n) \to 0$, then H is a compact connected set.*

Definition 1.2. *Let (X, d) be a metric space. For a function*

$$f : X \longrightarrow X$$

*let us denote by $Lip(f) \in [0, +\infty]$ the **Lipschitz constant** associated to f, which is $Lip(f) = \sup_{x,y\in X;\ ;x\neq y} \frac{d(f(x),f(y))}{d(x,y)}$. We say that f is a **Lipschitz function** if $Lip(f) < +\infty$ and a **contraction** if $Lip(f) < 1$.*

Definition 1.3.

a) *An **iterated function system (IFS)** on a metric space (X, d) consists in a finite family of contractions $(f_k)_{k=\overline{1,n}}$ on X and it is denoted by*

$$\mathcal{S} = (X, (f_k)_{k=\overline{1,n}}).$$

b) *A family of continuous functions $(f_i)_{i\in I}$, $f_i : X \longrightarrow X$ for every $i \in I$, is said to be **bounded** if for every bounded set $A \subset X$ the set $\cup_{i\in I} f_i(A)$ is bounded. An **infinite iterated function system (IIFS)** on X consists of a bounded family of continuous functions $(f_i)_{i\in I}$ on X and it is denoted by $\mathcal{S} = (X, (f_i)_{i\in I})$.*

Definition 1.4.

a) *For an (IFS), $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, the function $F_{\mathcal{S}} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$ defined by $F_{\mathcal{S}}(B) = \cup_{k=1}^{n} f_k(B)$ is called the **fractal operator** associated with the (IFS) $\mathcal{S}$.*

b) *For an (IIFS), $\mathcal{S} = (X, (f_i)_{i\in I})$, the function $F_{\mathcal{S}} : \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ defined by $F_{\mathcal{S}}(B) = \overline{\cup_{i\in I} f_i(B)}$ for every $B \in \mathcal{B}(X)$ is called the **fractal operator** associated with the (IIFS) $\mathcal{S}$.*

Remark 1.1. We remark that $Lip(F_{\mathcal{S}}) \leq \max_{k=\overline{1,n}} Lip(f_k)$ in the case of an (IFS) and $Lip(F_{\mathcal{S}}) \leq \sup_{i\in I} Lip(f_i)$ in the case of an (IIFS).

Using Banach's theorem there exists, for an (IFS) $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, a unique set A such that $F_{\mathcal{S}}(A) = A$, which is called the **attractor** of the (IFS) $\mathcal{S}$. More precisely we have the following well-known result.

Theorem 1.1. ([8]) *Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, an (IFS) with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$. Then there exists a unique set $A \in \mathcal{K}(X)$ such that $F_{\mathcal{S}}(A) = A$. Moreover, for any $H_0 \in \mathcal{K}(X)$ the sequence $(H_n)_{n\geq 1}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n)$ is convergent to A. For the speed of the convergence we have the following estimation:*

$$h(H_n, A) \leq \frac{c^n}{1-c} \cdot h(H_0, H_1).$$

For the infinite case, using again Banach's theorem, one can prove the following:

Theorem 1.2. ([11], [12], [16]) *Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_i)_{i \in I})$ an infinite iterated function system such that*

$$c = \sup_{i \in I} Lip(f_i) < 1.$$

Then there exists a unique set $A \in \mathcal{B}(X)$ such that $F_{\mathcal{S}}(A) = A$. The unique set $A \in \mathcal{B}(X)$ is called the attractor of the infinte iterated function system. Moreover, for any $H_0 \in \mathcal{B}(X)$ the sequence $(H_n)_{n \geq 0}$ defined by $H_{n+1} = F_{\mathcal{S}}(H_n)$ is convergent to A. For the speed of the convergence we have the following estimation:

$$h(H_n, A) \leq \frac{c^n}{1-c} h(H_0, H_1).$$

A generalization of the notion of (IFS) was given in ([17]), where topological iterated function systems (TIFS) were introduced. In ([17]) was also proven the existence and uniqueness of the attractor of a (TIFS).

Definition 1.5. ([17]) *A **topological iterated function system** consists on a finite family of continuous functions $\{f_j\}_{j=\overline{1,n}}$ defined on a topological separated space (X, τ) with the following properties:*

a) *For every compact subset $A \subset X$ there is a compact subset $B \subset X$ such that $A \subset B$ and $\cup_{j=1}^{n} f_j(B) \subset B$.*

b) *For every compact subset $A \subset X$ such that*

$$\bigcup_{j=1}^{n} f_j(A) \subset A,$$

the intersection

$$\bigcap_{n \geq 1} f_{j_1 j_2 \ldots j_n}(A)$$

is formed by one point for every $j_1, j_2, \ldots, j_n \in \{1, 2, \ldots, n\}$.

In the followings we briefly present the shift space of an (IFS). For more details one can see ([9]). We start with some set notations: $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{N}_n^* = \{1, 2, \ldots, n\}$. For two nonempty sets A and B, B^A denotes the set of functions from A to B. By $\Lambda = \Lambda(B)$ we will understand the set $B^{\mathbb{N}^*}$ and by $\Lambda_n = \Lambda_n(B)$ we will understand the set $B^{\mathbb{N}_n^*}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbb{N}^*}$ will be written as infinite words

$$\omega = \omega_1 \omega_2 \ldots \omega_m \omega_{m+1} \ldots,$$

where $\omega_m \in B$ and the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}_n^*}$ will be written as finite words $\omega = \omega_1 \omega_2 \ldots \omega_n$. By λ we will understand the empty word. Let us remark that $\Lambda_0(B) = \{\lambda\}$. By $\Lambda^* = \Lambda^*(B)$ we will understand the set of all finite words $\Lambda^* = \Lambda^*(B) = \cup_{n\geq 0}\Lambda_n(B)$. We denote by $|\omega|$ the length of the word ω. An element of $\Lambda = \Lambda(B)$ is said to have length $+\infty$. On the space $\Lambda = \Lambda(\mathbb{N}_n^*) = (\mathbb{N}_n^*)^{\mathbb{N}^*}$, we can consider the metric

$$d_s(\alpha,\beta) = \sum_{k=1}^{\infty} \frac{1-\delta_{\alpha_k}^{\beta_k}}{3^k}$$

where

$$\delta_x^y = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases},$$

$\alpha = \alpha_1\alpha_2\ldots$ and $\beta = \beta_1\beta_2\ldots$. Then $(\Lambda(\mathbb{N}_n^*), d_s)$ is called the code space or the shift space and it is a compact space. Let (X,d) be a complete metric space, $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$ an (IFS) on X and A the attractor of the $\mathcal{S}$. For any $\omega = \omega_1\omega_2\ldots\omega_m \in \Lambda_m(\mathbb{N}_n^*)$, f_ω will denote $f_{\omega_1}\circ f_{\omega_2}\circ\ldots\circ f_{\omega_m}$ and H_ω will denote $f_\omega(H)$ for every subset $H \subset X$. By H_λ we will understand the set H. In particular $A_\omega = f_\omega(A)$.

2. CONNECTEDNESS PROPERTIES OF THE ATTRACTORS

In this section we will give some results concerning the connectedness-type properties of an attractor. We will consider that the notions of connectedness, arcwise connectedness, locally arcwise connectedness and totally disconnectedness are well-known. We only remember the following definition.

Definition 2.1. *Let X be a nonempty set and $(A_i)_{i\in I}$ a family of nonempty subsets of X. Then the family $(A_i)_{i\in I}$ is called **connected** if for every $i,j \in I$ there exists $(i_k)_{k=\overline{1,n}} \subset I$ such that $i_1 = i$, $i_n = j$ and $A_{i_k}\cap A_{i_{k+1}} \neq \emptyset$ for every $k \in \{1,...,n-1\}$.*

In what concerns the totally disconnectedness of an attractor we have the following result:

Theorem 2.1. ([21]) *Let (X,d) be a complete metric space and $\mathcal{S} = (X,\{f_k\}_{k=\overline{1,n}})$ an (IFS) with $c = \max_{i=\overline{1,n}} Lip(f_i) < 1$. We also consider A the attractor of $\mathcal{S}$ and we denote $A_i = f_i(A)$ for every $i \in \{1,\ldots,n\}$. If $\sum_{i=1}^n Lip(f_i) < 1$, then A is **totally disconnected**.*

Remark 2.1. We remark that theorem 2.1 is not valid for (IIFSs). For that we will consider a countable iterated function system indexed by the rational numbers of the interval $[0,1]$. So, let $f_i : [0,1] \longrightarrow [0,1]$, $f_i(x) = i$ for every $x \in [0,1]$ and $i \in \mathbb{Q}\cap[0,1]$. Then the attractor of the countable iterated function system $\mathcal{S} = ([0,1],\{f_i\}_{i\in\mathbb{Q}\cap[0,1]})$ is the interval $[0,1]$, since

$$\overline{\bigcap_{i\in\mathbb{Q}\cap[0,1]} f_i([0,1])} = \overline{\bigcup_{i\in\mathbb{Q}\cap[0,1]} \{i\}} = \overline{\mathbb{Q}\bigcap[0,1]} = [0,1]$$

and we remark that $Lip(f_i) = 0$ for every $i \in \mathbb{Q} \cap [0,1]$. Hence

$$\sum_{i \in \mathbb{Q} \cap [0,1]} Lip(f_i) = 0 < 1,$$

but $[0,1]$ is not totally disconnected.

In what concerns the connectedness of the attractor of an (IFS) with have the following result:

Theorem 2.2. ([7], [9]) *Let (X,d) be a complete metric space and $\mathcal{S} = (X,(f_k)_{k=\overline{1,n}})$ an (IFS) with $c = \max_{i=\overline{1,n}} Lip(f_i) < 1$. We also consider A the attractor of $\mathcal{S}$ and we denote by $A_i = f_i(A)$ for every $i \in \{1,\ldots,n\}$. Then we have the equivalence:*

1) *The family $(A_i)_{i=\overline{1,n}}$ is connected.*

2) *A is arcwise connected.*

3) *A is connected.*

Remark 2.2.

a) Let (X,d) be a complete metric space and $A \in \mathcal{B}(X)$. For an element $a \in X$, f_a will denote the constant function of values a, that is $f_a : X \longrightarrow X$ and $f_a(x) = a$ for every $x \in X$. Then A is the attractor of the (IIFS) $\mathcal{S} = (X,(f_a)_{a \in A})$, if A is infinite or the (IFS) $\mathcal{S} = (X,(f_a)_{a \in A})$, if A is finite. Also A is the attractor of the (IIFS) $\mathcal{S}_B = (X,(f_a)_{a \in B})$, for any dense set B in A. In particular if A is separable and B is a countable dense set in A, then A is the attractor of the countable iterated function system $\mathcal{S}_B = (X,(f_a)_{a \in B})$. This happens, in particular for any compact set A. Because a closed (or compact) set could be connected but not arcwise connected it follows that the conditions 2) and 3) from theorem 2.2 are not equivalent for an (IIFS). Also, because the family of sets

$$(A_a = f_a(A) = \{a\})_{a \in A}$$

is not connected for every set A (in fact $A_a \cap A_b = \emptyset$) it follows that the points 1) and 2) from theorem 2.2 are not equivalent for an (IIFS). For more details one can see ([3]).

b) When one of the conditions of therem 2.2 is fulfilled, then the attractor of an (IFS) is a locally connected set. But it is not true, that every compact, connected and locally connected set can be the attractor of an (IFS). An example is given in ([10], [18]).

We will apply now the theorem 2.2. to the following well-known examples to find out whether the attractor is connected or not. For more details one can see ([1], [5], [6]).

Example 2.1. We consider the set $\mathbb{R}$ endowed with the distance given by the absolute value and the functions $f_1, f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f_1(x) = \frac{x}{2}$ and $f_2(x) = \frac{x+1}{2}$ and the (IFS) $\mathcal{S} = (\mathbb{R}, \{f_1, f_2\})$. Then the attractor$A = [0,1]$. Indeed, $F_{\mathcal{S}}([0,1]) = f_1([0,1]) \cup f_2([0,1]) = \left[0, \frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right] = [0,1]$. We have that $A_1 = f_1([0,1]) = \left[0, \frac{1}{2}\right]$, $A_2 = \left[\frac{1}{2}, 1\right]$ and $A_1 \cap A_2 = \left\{\frac{1}{2}\right\}$. We remark that A is connected and the family (A_1, A_2) is also connected.

Example 2.2. We consider the set $\mathbb{R}$ endowed with the distance given by the absolute value and the functions $f_1, f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f_1(x) = \frac{x}{3}$ and $f_2(x) = \frac{x+2}{3}$ and the (IFS)

$$\mathcal{S} = (\mathbb{R}, \{f_1, f_2\}).$$

Then the attractor $A = C \subset [0,1]$ is called the Cantor set. We have that $A_1 \subset \left[0, \frac{1}{3}\right]$, $A_2 \subset \left[\frac{2}{3}, 1\right]$ and $A_1 \cap A_2 = \emptyset$. Thus the family (A_1, A_2) is not connected and also $A = C$ is not a connected set. In fact, $A = C$ is totally disconnected.

Example 2.3. We consider the set $\mathbb{R}^2$ endowed with the euclidean distance and the functions $f_1, f_2, f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $f_1(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right)$, $f_2(x,y) = \left(\frac{x+1}{2}, \frac{y}{2}\right)$ and $f_3(x,y) = \left(\frac{x+1}{2}, \frac{y+1}{2}\right)$ and the (IFS)

$$\mathcal{S} = (\mathbb{R}, (f_1, f_2, f_3)).$$

The functions f_1, f_2, f_3 are similitudes of the plane with the coefficient $\frac{1}{2}$. The attractor A is called the Sierpinski triangle.

We have that $f_1(0,0) = (0,0)$, $f_2(1,0) = (1,0)$ and $f_3(0,1) = (0,1)$. So $(0,0), (1,0), (0,1) \in A$. Also

$$f_1(1,0) = f_2(0,0) = \left(\frac{1}{2}, 0\right) \in A_1 \cap A_2,$$

$$f_1(0,1) = f_3(0,0) = \left(0, \frac{1}{2}\right) \in A_1 \cap A_3,$$

$$f_2(0,1) = f_3(1,0) = \left(\frac{1}{2}, \frac{1}{2}\right) \in A_2 \cap A_3.$$

It results that the family of sets (A_1, A_2, A_3) is connected and thus A is connected and arcwise connected.

Next we wil study the case of the attrators with many connected components. We have a result similar to theorem 2.2.

Theorem 2.3. ([15]) *Let (X,d) be a complete metric space, $p \in \mathbb{N}^*$, $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ an (IFS) with $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$ and A the attractor of $\mathcal{S}$. Then we have the equivalence:*

1) *For every $\omega \in \Lambda_p = \Lambda_p(N_n^*)$ the family $(A_{\omega_i})_{i=\overline{1,n}}$ is connected.*

2) *The set A_ω is arcwise connected for every $\omega \in \Lambda_p$.*

3) *The set* A_ω *is connected for every* $\omega \in \Lambda_p$.

4) *The set* A_ω *is arcwise connected for every* $\omega \in \Lambda_m$ *and* $m \geq p$.

5) *The set* A_ω *is connected for every* $\omega \in \Lambda_m$ *and* $m \geq p$.

Moreover, if one of the conditions from 1)-5) is fullfilled then we have the following:

6) *A has at most* n^p *connected componenets, moreover, A has the same number of connected components as the family* $(A_\omega)_{\omega\in\Lambda_p}$.

7) *Each connected component of A is arcwise connected.*

8) *A is locally arcwise connected.*

Point 7) of theorem 2.3 was generalized in ([2]), in the sense that it is always true when the attractor has a finite number of connected components, without considering connectedness properties upon the sets A_ω, $\omega \in \Lambda_p$. Thus we have the following result:

Theorem 2.4. ([2]) *Let* (X, d) *be a complete metric space,* $\mathcal{S} = (X, (f_i)_{i=\overline{1,n}})$ *an (IFS) with* $c = \max_{k=\overline{1,n}} Lip(f_k) < 1$ *and A the attractor of* $\mathcal{S}$. *If A has a finite number of connected components, then each connected component is arcwise connected.*

Example 2.4. ([15]) We consider the function $\phi : [0, 1] \longrightarrow [0, 1]$ defined by

$$\phi(x) = \begin{cases} \frac{x}{2} & \text{if } x \in \left[0, \frac{1}{4}\right] \\ \frac{1}{8} & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ \frac{x}{2} - \frac{1}{4} & \text{if } x \in \left[\frac{3}{4}, 1\right] \end{cases}.$$

Then $Lip(\phi) = \frac{1}{2}$. Let $\psi : [0, 1] \longrightarrow [0, 1]$ a function defined by $\psi(x) = 1 - \phi(1 - x)$. Then $Lip(\psi) = \frac{1}{2}$. We consider the iterated function system $\mathcal{S} = ([0, 1], \{\phi, \psi\})$. Then $A = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$. Indeed,

$$\phi\left(\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]\right) = \phi\left(\left[0, \frac{1}{4}\right]\right) \cup \phi\left(\left[\frac{3}{4}, 1\right]\right) == \left[0, \frac{1}{8}\right] \cup \left[\frac{1}{8}, \frac{1}{4}\right] = \left[0, \frac{1}{4}\right]$$

and

$$\psi\left(\left[0, \tfrac{1}{4}\right] \cup \left[\tfrac{3}{4}, 1\right]\right) = \psi\left(\left[0, \tfrac{1}{4}\right]\right) \cup \psi\left(\left[\tfrac{3}{4}, 1\right]\right) =$$

$$= \left(1 - \phi\left(\left[\tfrac{3}{4}, 1\right]\right)\right) \cup \left(1 - \phi\left(\left[0, \tfrac{1}{4}\right]\right)\right) = \left[\tfrac{3}{4}, \tfrac{7}{8}\right] \cup \left[\tfrac{7}{8}, 1\right] = \left[\tfrac{3}{4}, 1\right].$$

Thus

$$F_{\mathcal{S}}\left(\left[0, \tfrac{1}{4}\right] \cup \left[\tfrac{3}{4}, 1\right]\right) = \phi\left(\left[0, \tfrac{1}{4}\right] \cup \left[\tfrac{3}{4}, 1\right]\right) \cup \psi\left(\left[0, \tfrac{1}{4}\right] \cup \left[\tfrac{3}{4}, 1\right]\right) =$$

$$= \left[0, \tfrac{1}{4}\right] \cup \left[\tfrac{3}{4}, 1\right].$$

It results that $A = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$. We have that $A_1 = \phi(A) = \left[0, \frac{1}{4}\right]$, $A_2 = \psi(A) = \left[\frac{3}{4}, 1\right]$, $A_{11} = \left[0, \frac{1}{8}\right]$, $A_{12} = \left[\frac{1}{8}, \frac{1}{4}\right]$, $A_{21} = \left[\frac{3}{4}, \frac{7}{8}\right]$ and $A_{22} = \left[\frac{7}{8}, 1\right]$. Then $A_1 \cap A_2 = \emptyset$, $A_{11} \cap A_{12} = \left\{\frac{1}{8}\right\}$ and $A_{21} \cap A_{22} = \left\{\frac{7}{8}\right\}$. That means the families of set (A_{11}, A_{12}) and (A_{21}, A_{22}) are connected. We remark that A_1 and A_2 are connected sets, but A is not connected.

Remark 2.3. Theorem 2.4 does not remain true if the attractor has an infinite number of connected components. An example can be found in ([2]).

We will now study the case of an (IIFS). We give some sufficient conditions for an attractor of (IIFS) to be connected. (IIFS) were introduces in ([20]). More results and examples can be found in ([3], [12], [13], [16]).

Theorem 2.5. ([3]) *Let (X, d) be a complete metric space,*

$$\mathcal{S} = (X, (f_i)_{i \in I})$$

an (IIFS) and A the attractor of $\mathcal{S}$. Let $I_j \subset I$ for every $j \in J$ such that:

1) $I = \cup_{j \in J} I_j$,

2) $\cup_{j \in J} B_j$ *is a connected set, where $B_j := A(\mathcal{S}_j)$ is the attractor of*

$$\mathcal{S}_j = (X, (f_i)_{i \in I_j})$$

   for every $j \in J$.

Then A is a connected set.

The folowing are corollaries of theorem 2.5.

Corolarry 2.1. ([3]) *Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_i)_{i \in I})$ an (IIFS). Let $I_j \subset I$ for every $j \in J$ such that:*

1) $I = \cup_{j \in J} I_j$.

2) *B_j is connected, where $B_j := A(\mathcal{S}_j)$ is the attractor of $\mathcal{S}_j = (X, (f_i)_{i \in I_j})$ for every $j \in J$.*

3) *The family $(B_j)_{j \in J}$ is connected.*

Then A is connected.

Corolarry 2.2. ([3]) *Let (X, d) be a complete metric space and $\mathcal{S} = (X, (f_i)_{i \in I})$ an (IIFS). Let $I_j \subset I$ for every $j \in J$ such that:*

1) $I = \cup_{j \in J} I_j$.

2) *I_j is finite for every $j \in J$.*

3) *The families of sets $(f_i(B_j))_{i \in I_j}$ are connected, where $B_j := A(\mathcal{S}_j)$ is the attractor of $\mathcal{S}_j = (X, (f_i)_{i \in I_j})$ for every $j \in J$.*

4) *The family$(B_j)_{j \in J}$ is connected.*

Then A is connected.

Concerning the attractors of (TIFSs), it is proven in ([4]) that the nonempty union of a Peano space and a segment is the attractor of a (TIFS) formed by three functions. Examples can be found in ([4], [17]).

Theorem 2.6. ([4]) *We consider a topological separated space (X, τ) and $t : [0,1] \longrightarrow X$ a continuous function. Then $K = t([0,1])$ is a Peano space. Let $f : [0,1] \longrightarrow X$ be a continuous and injective function. We consider now the space $H = K \cup f([0,1])$, where $K \cap f([0,1]) = \{f(0)\}$ and, moreover, we suppose that $t(0) = f(0)$. Then H is the attractor of a topological iterated function system formed by three functions.*

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