On the Exponential Diophantine Equation $x^2 + D = y^n$: a brief survey

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Abstract
We give a survey on some important results on the exponential Diophantine equation $x^2 + D = y^2$.

Keywords: *exponential Diophantine equation, Lebesgue-Nagell equation, generalized Ramanujan-Nagell equation*

ACM/AMS Classification: 11D41, 11D61.

1. Introduction

In this paper, we focus our attention to the equation

$$x^2 + D = y^n, \ \textit{in integers } x, y, n \geq 3 \tag{1}$$

where D is a positive integer. We present however, for some particular cases, solutions with $x = 1$ (i.e. $x < 3$).

V. A. Lebesgue [35] proved in 1850 that there are no non-trivial solutions for $D = 1$. Nagell [44] proved in 1923 that equation (1) has no solutions for $D = 3$ and $D = 5$. Because Lebesgue and Nagell were the first mathematicians with concrete results concerning equation (1), this equation is called in [17] the Lebesgue-Nagell equation.

S.Ramanujan [53] asked in 1913 if the Diophantine equation $x^2 + 7 = 2^n$ had any positive solutions (x, n) other than $(1, 3), (3, 4), (5, 5), (11, 7)$ and $(181, 15)$. Nagell [45] ([48] in English) proved in 1948 that these are the only solutions. That's why equation $x^2 + 7 = 2^n$ is often called the Ramanujan-Nagell equation. Cohen [28] made a survey of its history and related problems. Ribenboim collected Nagell's works in [49].

A comprehensive survey on equation (1) is given by Abu Muriefah and Bugeaud [1]. We complete that survey with recent results, especially when D is in some infinite set (see section 3 of the present survey).

2. **The Diophantine equation $x^2 + D = y^n$, where D is fixed**

As mentioned in section 1, equation (1) was solved by Lebesgue for $D = 1$ and by Nagell for $D = 3$ and $D = 5$. The case $D = 3$ was also proved by Brown [16], and then by Cohn [26].

Ljunggren [38] solved (1) for $D = 2$, finding the only solution $5^2+2 = 3^3$. Cohn asserted in [25] that Euler found the same solution for $D = 2$ in [31]. Nagell [46] also gave the solution for $D = 2$. Nagell [47] solved the case $D = 4$, obtaining the only solutions $2^2 + 4 = 2^3$ and $11^2 + 4 = 5^3$. A more elementary proof for this case was given by Sury [58].

Cohn [25] completed the solutions for 77 values of D, where $1 \leq D \leq 100$, using elementary methods. He established that there are no solutions at all for $D \in \{1, 3, 5, 6, 8, 9, 10, 14, 21, 22, 24, 27, 29, 30, 33, 34, 36, 37, 38, 41, 42, 43, 46, 50, 51, 52, 57, 58, 59, 62, 66, 68, 69, 70, 73, 75, 78, 82, 84, 85, 88, 90, 91, 93, 94, 98\}$. He also gave solutions for 31 values of D (see Table 1):

Mignotte and de Weger [43] solved equation (1) for $D = 74$, obtaining $(x, y, n) =$ (13,3,5), (985,99,3) and proved that equation (1) has no solution for $D = 86$.

Bennett and Skinner [12] applied theory of Galois representations and modular forms to solve the case $D = 55$, obtaining $(x, y, n) =$ (3,2,6), (3,4,3), (419,56,3) and the case $D = 95$, obtaining (x, y, n)=(11,6,3),(529,6,7).

The remaining values for D were solved in 2004 by Bugeaud, Mignotte and Siksek [17] (see Table 2).

3. **The Diophantine equation $x^2 + D = y^n$, with D in some infinite set**

In recent years, equation (1) has been analyzed also in the more general case when D is not fixed but $D \in S$ with $D > 0$. One major result, called the 'Theorem BHV', was obtained in [15] by Bilu, Hanrot and Voutier, who completely solved the problem of existence of primitive divisors in Lucas-Lehmer sequences. This theorem has many applications to Diophantine equations and it was applied in some papers mentioned below.

Cohn [24] proved that if $D = 2^{2k+1}$, then equation (1) has solutions (three families of solutions) only when n = 3.

Arif and Abu Muriefah [7] conjectured that if $D = 2^k$, the only solutions are then given by $(x; y) = (2^k;\ 2^{2k+1})$ and $(x; y) = (11^{2k-1};\ 5 \cdot 2^{2(k-1)/3})$, the latter solution existing only when $(k; n) = (3M+1; 3)$ for some integer $M \geq 0$. Arif and Abu Muriefah obtained partial results towards this conjecture in [7] and also did Cohn in [27]. Arif and Abu Muriefah finally proved the conjecture in [9]. Le [34] and Siksek [55] gave alternative proofs.

Abu Muriefah and Arif [3] conjectured that "there are no solutions for the Diophantine equation $x^2 + 3^{2k} = y^n$, where $n \geq 3$ unless $k = 3K + 2$ and $n = 3$ and then there is a unique solution $x = 46 \cdot 3^{3K}$ and $y = 13 \cdot 32^{K}$". Luca proved this conjecture in [39].

It was proved by Arif and Abu Muriefah in [8] that equation (1) has exactly one (infinite) family of solutions if $D = 3^{2k+1}$. Luca [39] solved the case $D = 3^{2k}$ if $gcd(x, y) = 1$. Liqun [36] solved the equation $x^2 + 3^m = y^n$ for both odd and even m.

The case $D = 2^a 3^b$ (a and b being arbitrary non-negative integers) and $gcd(x, y) = 1$, was completely solved by Luca [40].

The case $D = 5^{2k}$ has been considered by Arif and Abu Muriefah in [6], who established that equation (1) may have a solution only if 5 divides x and p does not divide k for any odd prime p dividing n. The same authors proved in [4] that if $D = 5^{2k+1}$, then equation (1) has no solutions for all $k \geq 0$. Several results has been also obtained by Abu Muriefah and Arif in [2] for $D = q^{2k}$, where q is an odd prime. The same equation is independently solved by Liqun in [37].

Sardha and Srinivasan [54] discussed equation (1) for $D = p_1^{\alpha_1}...p_r^{\alpha_r} = D_s D_t^2$, where $p_1, ..., p_r$ are primes, $\alpha_1, ..., \alpha_r$ are positive integers and D_s is the square free part of D. They gave many examples for D with $D_s \leq 10000$.

Bérczes and Pink [14] investigated equation $x^2 + d^{2l+1} = y^n$ in unknown integers (x, y, l, n) with $x \geq 1$, $y \geq 1$, $n \geq 3$, $l \geq 0$ and $gcd(x, y) = 1$. They extended the result of Saradha and Srinivasan [54] to the case $h(-d) \in \{2, 3\}$, where $d > 0$ is a squarefree integer and $h = h(-d)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

Pink [51] studied the case $D = 2^a 3^b 5^c 7^d$ with $gcd(x, y) = 1$, where a, b, c, d are non-negative integers.

Luca and Togbé discussed equation (1) for $D = 7^{2k}$ [41] and for $D = 2^a 5^b$ [42].

The case $D = 2^a 5^b 13^c$ was studied by Goins, Luca, and Togbé [32]. The case $D = 5^a 13^b$ was treated in [5] by Abu Muriefah, Luca and Togbé.

Arif and Abu Muriefah [10] determined all the solutions of equation $x^2 + q^{2k+1} = y^n$, with $q \geq 5$ an odd prime, $q \not\equiv 7 (mod\, 8)$ and $gcd(n, 3h_0) = 1$ and $n \geq 3$, h_0 denoting the class number of the field $\mathbb{Q}(\sqrt{-q})$.

Le [33] gave all the solutions of equation (1) in the particular case when $gcd(x, y) = 1$, $D = p^2$, p prime with $3 \leq p < 100$. Tengely [59] completely solved (1) for $D = a^2$ with $3 \leq a \leq 501$ and a odd, under the assumption $(x, y) \in \mathbb{N}^2$, $gcd(x, y) = 1$.

The equation $A^4 + B^2 = C^n$ for $AB \neq 0$ and $n \geq 4$ was completely solved by Bennett, Ellenberg, and Nathan [11]. Ellenberg also treated this equation in [30].

Bérczes and Pink [13] completely solved the equation $x^2 + p^{2k} = y^n$, where $2 \leq p < 100$ is a rational prime and integer unknowns x, y, n, k satisfy $x \geq 1, y > 1, n \geq 3$ prime, $k \geq 0$ and $gcd(x, y) = 1$. They also established, as a corollary, that there are no solutions to the equation $x^2 + p^{2k} = y^p$ in integer unknowns (x, y, p, k) with $x \geq 1$, $y > 1$, $p \geq 5$ prime, $k \geq 0$ and $gcd(x, y) = 1$.

Cenberci and Senay [22] established that if $y \equiv 5 (mod\, 8)$ is a prime power, then the conjecture "if $a^2 + B^2 = y^4$ with $gcd(a, B, y) = 1$ and a even, and (a, B, y^2) is a Pythagorean triples then the Diophantine equation $x^2 + B^m = y^n$ has the only positive integral solution $(x, m, n) = (a, 2, 4)$" holds (and also Terai conjecture, presented in [60], holds).

Cenberci and Senay [23] discussed the equation $x^2 + q^m = p^n$, in relation with Terai conjecture, with p and q odd primes, which satisfy $q^2 + 1 = 2p^2$ and other conditions. They also gave all solutions for five examples with b and c primes, such that $b^2 + 1 = 2c^2$, $b < 20.000$ and $c < 157.000$.

Zhu and Le [63] gave all solutions of some generalized Lebesque- Nagell equations $x^2 + q^m = y^n$, where the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$ is one.

Zhu discussed in [62] equation $x^2 + q^m = y^3$.

Demirpolat, Cenberci and Senay [29] established that the Diophantine equation $x^2 + 11^{2k+1} = y^n$ has exactly only one family of solution, when n is an odd integer, $n \geq 3$, $k \geq 0$, and $h = 1$ is the class number of the field $\mathbb{Q}(\sqrt{-11})$.

Cangül, Soydan and Simsek [20] found all solutions of Diophantine equation $x^2 + 11^{2k} = y^n, x \geq 1, y \geq 1, k \in \mathbb{N}, n \geq 3$ and gave p-adic interpretation of that equation.

Cangül, Demirci, Luca, Pinter and Soydan treated in [18] equation (1) for $D = 2^a 11^b$ and gave the complete solution (n, x, y) with $n \geq 3$ and $gcd(x, y) = 1$. Cangül, Demirci, Inam, Luca and Soydan [21] discussed equation (1) for $D = 2^a 3^b 11^c$ and gave the complete solution (n, x, y) with $n \geq 3$ and $gcd(x, y) = 1$.

The complete solution (n, a, b, x, y) of the equation $x^2 + 5^a 11^b = y^n$ when $gcd(x, y) = 1$, except for the case when xab is odd, has been obtained by Cangül, Demirci, Soydan and Tzanakis in [19].

Pink and Rabái [52] gave all the solutions to equation $x^2 + 5^k 17^l = y^n$ in unknown integers $(x; y; k; l; n)$ with $x \geq 1$, $y > 1$, $n \geq 3$, $k \geq 0$, $l \geq 0$ and $gcd(x; y) = 1$.

Soydan, Ulas and Zhu [56] completely solved the equation $x^2 + 2^a 19^b = y^n$, where $x \geq 1$, $y > 1$, $n \geq 3$, $a, b \geq 0$, $l \geq 0$ and $gcd(x; y) = 1$.

Soydan [57] gave all the solutions to equation $x^2 + 7^\alpha 11^\beta = y^n$ for the non-negative integers $\alpha; \beta; x; y; n \geq 3$, where x and y co-prime, except when α, x is odd and β is even.

Peker and Cenberci [50] completely solved equation $x^2 + 19^m = y^n$, by treating the equation for m even and odd separately.

Xiaowei [61] gave a complete classification of all positive integer solutions (x, y, m, n) of the equation $x^2 + p^{2m} = y^n$, $gcd(x, y) = 1, n > 2$, where p is an odd prime and solved the equation for certain interesting cases.

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Table 1: Cohn's solutions

$D = 2$	(x, y, n) = (5,3,3)
$D = 4$	(x, y, n) = (2,2,3), (11,5,3)
$D = 11$	(x, y, n) = (4,3,3), (58,15,3)
$D = 12$	(x, y, n) = (2,2,4)
$D = 13$	(x, y, n) = (70,17,3)
$D = 16$	(x, y, n) = (4,2,5)
$D = 17$	(x, y, n) = (8,3,4)
$D = 19$	(x, y, n) = (18,7,3), (22434,55,5)
$D = 20$	(x, y, n) = (14,6,3)
$D = 26$	(x, y, n) = (1,3,3), (207,35,3)
$D = 32$	(x, y, n) = (7,3,4), (88,6,5)
$D = 35$	(x, y, n) = (36,11,3)
$D = 40$	(x, y, n) = (52,14,3)
$D = 44$	(x, y, n) = (9,5,3)
$D = 48$	(x, y, n) = (4,4,3), (4,2,6), (148,28,3)
$D = 49$	(x, y, n) = (24,5,4), (524,65,3)
$D = 53$	(x, y, n) = (26,9,3), (26,3,6), (156,29,3)
$D = 54$	(x, y, n) = (17,7,3)
$D = 56$	(x, y, n) = (5,3,4), (76,18,3)
$D = 61$	(x, y, n) = (8,5,3)
$D = 64$	(x, y, n) = (8,2,7)
$D = 65$	(x, y, n) = (4,3,4)
$D = 67$	(x, y, n) = (110,23,3)
$D = 76$	(x, y, n) = (7,5,3), (1015,101,3)
$D = 77$	(x, y, n) = (2,3,4)
$D = 80$	(x, y, n) = (1,3,4)
$D = 81$	(x, y, n) = (46,13,3)
$D = 83$	(x, y, n) = (140,27,3), (140,3,9)
$D = 89$	(x, y, n) = (6,5,3)
$D = 96$	(x, y, n) = (23,5,4)
$D = 97$	(x, y, n) = (48,7,4)

Table 2: Bugeaud, Mignotte and Siksek's solutions

$D = 7$	(x, y, n) = (1,2,3), (181,32,3), (3,2,4), (5,2,5), (181,8,5)
$D = 15$	(x, y, n) = (7,4,3), (1,2,4), (7,2,6)
$D = 18$	(x, y, n) = (3,3,3), (15,3,5)
$D = 23$	(x, y, n) = (2,3,3), (3,2,5), (45,2,11)
$D = 25$	(x, y, n) = (10,5,3)
$D = 28$	(x, y, n) = (6,4,3), (22,8,3), (225,37,3), (2,2,5), (6,2,6), (10,2,7), (22,2,9), (362,2,17)
$D = 31$	(x, y, n) = (15,4,4), (1,2,5), (15,2,8)
$D = 39$	(x, y, n) = (5,4,3), (31,10,3), (103,22,3), (5,2,6)
$D = 45$	(x, y, n) = (96,21,3), (6,3,4)
$D = 47$	(x, y, n) = (13,6,3), (41,12,3), (500,63,3), (14,3,5), (9,2,7)
$D = 60$	(x, y, n) = (2,4,3), (1586,136,3), (14,4,4), (50354,76,5), (2,2,6), (14,2,8)
$D = 63$	(x, y, n) = (1,4,3), (13537,568,3), (31,4,5), (1,2,6), (31,2,10)
$D = 71$	(x, y, n) = (21,8,3), (35,6,4), (46,3,7), (21,2,9)
$D = 72$	(x, y, n) = (12,6,3), (3,3,4)
$D = 79$	(x, y, n) = (89,20,3), (7,2,7)
$D = 87$	(x, y, n) = (16,7,3), (13,4,4), (13,2,8)
$D = 92$	(x, y, n) = (6,2,7), (90,2,13)
$D = 99$	(x, y, n) = (12,3,5)
$D = 100$	(x, y, n) = (5,5,3), (30,10,3), (198,34,3), (55,5,5)