A geometric proof of a non-vanishing theorem on Del Pezzo variety V^n

Alexandru Sterian
Faculty of Mathematics and Informatics
Spiru Haret University
Bucharest, Romania
alexandru.sterian@gmail.com

Abstract
In this paper, we give a new proof of a non-vanishing result from [4], concerning first homology group of a line bundle defined on Del Pezzo toric variety of type V^n.
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1. Introduction

In [4], we construct indecomposable rank-2 vector bundles on an n dimensional Del Pezzo variety V^n. In fact, we show that for every integer $t \geq 1$ and $p \in \{1, ..., n\}$, any non trivial rank-2 vector bundle $\mathcal{E}$, definied on an n-dimensional Del Pezzo variety of type V^n, by the following exact sequence

$$0 \to \mathcal{O}_{V^n}(-tE_p - tE_{2n+2}) \to \mathcal{E} \to \mathcal{O}_{V^n}(tE_p + tE_{2n+2}) \to 0,$$

is indecomposable. Our main tool is the following result:

Theorem 1.1. *([4]) Let V^n be a $n > 2$ dimensional Del Pezzo variety, $t \geq 1$ an integer and E_i the toric divisors corresponding to the ray generators u_i of V^n, for $i \in \{1, ..., 2n+2\}$.*

Then for every $p \in \{1, ..., n\}$, we have the following non-vanishing result:

$$H^1(V^n, \mathcal{O}(-2tE_p - 2tE_{2n+2})) = 2t - 1.$$

In [4], we provide a proof of this theorem using combinatorial and topological methods. In this note, we give a geometric proof of this result based on the vanishing theorem of Batyrev and Borisov.

2. About Del Pezzo toric variety of type V^n

In this paper we consider $n = 2r \geq 4$ an even integer.

Definition 2.1. *The n-dimensional Del Pezzo toric variety V^n is the smooth Fano toric variety whose ray generators are:*

$$u_i = e_i, for\, i = 1 \dots n$$
$$u_{n+1} = -e_1 - \dots - e_n$$
$$u_{i+n+1} = -e_i, for\, i = 1 \dots n$$
$$u_{2n+2} = e_1 + \dots + e_n,$$

where $e_1, .., e_n$ is a basis of M, which we will identify with $\mathbb{Z}^n$.

Starting from the ray generators $u_1, ..., u_{2n+2}$ in Σ_{V^n}, split them in two subsets:

$$x_i = u_i, \text{ for } 1 \leq i \leq n+1,$$

and

$$y_j = u_{j+n+1}, \text{ for } 0 \leq j \leq n+1.$$

The fan of Del Pezzo variety V^n (denoted Σ_{V^n}) is defined by prescribing that the rays y_i and x_i can not appear at the same time in any cone that Σ_{V^n} contains as part of its support (see [2]). More precisely, for $n = 2r$, the fan Σ_{V^n} is the union of the $\Sigma_{V^n}(m)$ for $0 \leq m \leq n$, where:

$$\Sigma_{V^n}(m) = \{\langle x_i, y_j \rangle_{i \in I, j \in J} | I, J \subset \{1, 2, ..., n+1\}, I \cap J = \emptyset,$$
$$\#I \leq r, \#J \leq r, \#I \cup J = m\}.$$

In this note, we denote with E_i the toric divisors associated to the ray generators u_i of V^n.

Next, we recall some basic notions and theorems that will need in our proof, the main source for this being [3]. Let $M = \mathbb{Z}^n$.

Definition 2.2. *(see [3]) Let $D = \sum_{k=1,...2n+2} a_k E_k$ be a Cartier divisor on V^n. The polytope associated to D, which we denote by P_D is the polytope in $\mathbb{R}^n = M \otimes_{\mathbb{Z}} \mathbb{R}$ defined by the following:*

$$P_D = \{x = (x_1, ..., x_n) \in \mathbb{R}^n \mid \langle x, u_k \rangle \geq -a_k, \forall k = 1, ..., 2n+2\},$$

where u_k are the ray generators of V^n.

In this case, $\langle ., . \rangle$ denotes the euclidian scalar product on $\mathbb{R}^n$. Since V^n is smooth, every divisor on V^n will be Cartier.

Proposition 2.1. *(see [3]) Let D a (Cartier) divisor on V^n. Then for all maximal cones $\sigma \in \Sigma_{V^n}(n)$, there exists a unique $m_\sigma \in M$ such that $\langle m_\sigma, u_k \rangle = -a_k$, for all ray generators u_k which appear in σ.*

Proof. See[3], Theorem 4.2.8 . $\square$

The next result provides an important characterization of base-point free divisors on V^n:

Proposition 2.2. *(see [3]) Let D a Cartier divisor on a Del Pezzo variety V^n. Then D is base-point free if and only if $m_\sigma \in P_D$ for all maximal cones $\sigma \in \Sigma_{V^n}(n)$.*

Proof. See[3], Proposition 6.1.1 . $\square$

Proposition 2.3. *Let V^n be a $n > 2$ dimensional Del Pezzo variety, $t \geq 1$ an integer and E_i the toric divisors corresponding to the ray generators u_i of V^n, for $i \in \{1, ..., 2n+2\}$.*

Then for every $p \in \{1, ..., n\}$, the divisor

$$D = 2tE_p + 2tE_{2n+2}$$

is base-point free.

Proof. The fact that the divisor $D = 2tE_p + 2tE_{2n+2}$ with $t \geq 1$ an integer is base-point free is a direct implication of Proposition 2.2.

First, we compute the polytop associated to the divisor D.

Using the definition 2.2, we find that the polytop corresponding to the divisor $D = 2tE_p + 2tE_{2n+2}$ is

$P_{2tE_p+2tE_{2n+2}} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that

$$\langle x, e_1 \rangle \geq 0$$

$$\ldots$$

$$\langle x, e_{p-1} \rangle \geq 0$$

$$\langle x, e_p \rangle \geq -2t$$

$$\langle x, e_{p+1} \rangle \geq 0$$

$$\ldots$$

$$\langle x, -e_1 - \ldots - e_n \rangle \geq 0$$

$$\langle x, -e_1 \rangle \geq 0$$

$$\ldots$$

$$\langle x, -e_n \rangle \geq 0$$

$$\langle x, e_1 + \ldots + e_n \rangle \geq -2t\}.$$

Thus, $P_{2tE_p+2tE_{2n+2}} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that

$$x_1 = ...x_{p-1} = x_{p+1} = ... = x_n = 0,$$

$$\text{and } 0 \geq x_p \geq -2t\}.$$

Obviously, $P_{2tE_p+2tE_{2n+2}}$ is convex.
Next, we determine the Cartier data m_σ corresponding to $D = 2tE_p+2tE_{2n+2}$. For this divisor, the only $a_k \neq 0$ are $a_p = 2t$ and $a_{2n+2} = 2t$. Moreover, we can easily show using Proposition 2.1, that for every maximal cone $\sigma \in \Sigma_{V^n}(n)$, $m_\sigma = -2te_p$, only if one of the ray generators $v_p = e_p$ or $v_{2n+2} = \sum_{i=1}^{n} e_i$ is contained in σ. For all the others maximals cone σ that don't contain v_p or v_{2n+2}, it follows $m_\sigma = 0_{\mathbb{R}^n}$.
As a consequence, a simple verification shows that for every $\sigma \in \Sigma_{V^n}(n)$, $m_\sigma \in P_{2tE_p+2tE_{2n+2}}$.
Therefore, the divisor $D = 2tE_p + 2tE_{2n+2}$ with $t \geq 1$ verifies the hypothesis of Proposition 2.2, and so is base-point free. □

For the proof of the main result of this note we will need the next theorem:

Theorem 2.1. ***(Batyrev-Borisov)****(see [3])*
Let D be a Cartier divisor on a toric variety X. Assume that D is nef then:

•$H^i(X, \mathcal{O}_X(-D)) = 0, \ \forall\, i \neq \dim P_D$.

•$H^i(X, \mathcal{O}_X(-D)) = \bigoplus_{m \in Relint(P_D)\cap M} \mathbb{C}.\chi^{-m}$*, for* $i = dimP_D$*, where* $Relint(P_D)$ *denotes the relative interior of the polytop* P_D*.*

Proof. See[3], Theorem 9.2.7 . □

To apply this result to our situation we only need the following lemma:

Lemma 2.1. *(see [3]) Let D be a Cartier divisor on a toric variety X such that the fan Σ_X has convex support. Then D is nef if and only if lD is base-point free for some integer $l > 0$.*

Proof. See[3], Lemma 9.2.1 . □

3. The Proof of Theorem 1.1

Proof. From Proposition 2.3, we saw that the divisor $D = 2tE_p + 2tE_{2n+2}$ is base-point free, so applying Lemma 2.1, we deduce that D is nef.
As a consequence, Theorem 2.1 implies

$$H^i(X, \mathcal{O}_{V^n}(-2tE_p - 2tE_{2n+2})) = 0 \text{ for all } i \neq \dim P_{2tE_p+2tE_{2n+2}}.$$

Since $P_{2tE_p+2tE_{2n+2}} = \{x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that

$$x_1 = ...x_{p-1} = x_{p+1} = ... = x_n = 0,$$

$$\text{and } 0 \geq x_p \geq -2t\},$$

it follows that $dimP_{2tE_p+2tE_{2n+2}} = 1$.
On the other side, $Relint(P_D) \cap M = \{-1, ..., -2t+1\}$.

Therefore, $H^1(V^n, \mathcal{O}_{V^n}(-2tE_p - 2tE_{2n+2})) = 2t-1$. $\square$

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