SOME UNIVALENCE CONDITIONS FOR A FAMILY OF INTEGRAL OPERATORS

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Abstract

In this paper, we consider two families of integral operators for analytic functions f_i, $i \in \{1, 2, ..., n\}$ in the open unit disk $\mathbb{U}$. The main object of the present paper is to discuss some univalence conditions for these integral operators. Many known univalence conditions are shown to follow upon specializing the parameters involved in our main results.

Keywords: *Analytic functions; Open Unit disk; Univalence conditions; General Scwarz Lemma; Integral operators.*

ACM/AMS Classification: Primary 30C45; Secondary 30C75.

1. **Introduction**

Let $\mathbb{U}$=$\{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in $\mathbb{U}$ and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$.
We begin by recalling a theorem dealing with univalence criterion, which will

be required in our present work.
In [6], Pascu gave the following univalence criterion for the functions $f \in \mathcal{A}$.
Theorem 1.1. [6] *Let $f \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. If* $\mathrm{Re}\beta > 0$ *and*

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \qquad (z \in \mathbb{U})$$

then the function

$$F_\beta(z) = \left(\beta\int_0^z u^{\beta-1}f'(u)du\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.
On the other hand, for the function $f \in \mathcal{A}$, Ozaki and Nunokawa [5] proved another univalence condition asserted by Theorem 1.2.
Theorem 1.2. [5] *Let $f \in \mathcal{A}$ satisfy the following condition*

$$\left|\frac{z^2 f'(z)}{[f(z)]^2} - 1\right| \leq 1 \qquad (z \in \mathbb{U}).$$

Then the function $f(z)$ is in the class $\mathcal{S}$.
In [9] is defined the class $\mathcal{S}(p)$. For $0 < p \leq 2$, let $\mathcal{S}(p)$ denote the class of function $f \in \mathcal{A}$ which satisfies the conditions

$$f(z) \neq 0 \qquad (0 < |z| < 1)$$

and

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p \qquad (z \in \mathbb{U}).$$

Singh [8] has shown that if $f(z) \in \mathcal{S}(p)$, then $f(z)$ satisfies

$$\left|\frac{z^2 f'(z)}{[f(z)]^2} - 1\right| \leq p\,|z|^2 \qquad (z \in \mathbb{U}). \tag{1}$$

Here, in our investigation, we consider two general families of integral operators. The first family of integral operators, studied by D. Breaz and N. Breaz [1], is defined as follows:

$$F_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}(z) = \left(\beta\int_0^z t^{\beta-1}\prod_{i=1}^n\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}}dt\right)^{\frac{1}{\beta}} \tag{2}$$

$$f_i \in \mathcal{A};\ \alpha_i, \beta \in \mathbb{C}\setminus\{0\} \text{ for all } i \in \{1,2,\dots,n\}.$$

Remark 1.3. For $n = 1$, the integral operator defined in (2) would reduce to the integral operator

$$F_{\alpha,\beta}(z) = \left(\beta\int_0^z t^{\beta-1}\left(\frac{f(t)}{t}\right)^{\frac{1}{\alpha}}dt\right)^{\frac{1}{\beta}}$$

studied in [7].
The second family of integral operators was introduced by D. Breaz and N. Breaz [2] and it has the following form:

$$G_{\alpha_1,\alpha_2,...,\alpha_n}(z) = \left(\left(\sum_{i=1}^{n}(\alpha_i - 1) + 1\right)\int_0^z \prod_{i=1}^{n}(f_i(t))^{\alpha_i - 1}\,dt\right)^{\frac{1}{\left(\sum_{i=1}^{n}(\alpha_i-1)+1\right)}} \tag{3}$$

$$f_i \in \mathcal{A};\ \alpha_i \in \mathbb{C} \text{ for all } i \in \{1, 2, ..., n\}.$$

Remark 1.4. For $n = 1$, the integral operator $G_\alpha(z)$ was studied by Moldoveanu and Pascu [3].
In the present paper, we study the univalence conditions involving the general families of integral operators defined by (2) and (3).
For this purpose, we need the following result.
Lemma 1.5. (General Schwarz Lemma) [4] *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m \qquad (z \in \mathbb{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

2. **Main Results**

Theorem 2.1. Let the functions $f_i \in \mathcal{S}(p_i)$ satisfy the inequality (1) with $0 < p_i \leq 2$ and $M_i \geq 1$ for all $i \in \{1, 2, ..., n\}$. Also, let α_i, β be complex numbers with the property $\text{Re}\beta > 0$.
If

$$|f_i(z)| \leq M_i \qquad (z \in \mathbb{U})$$

and

$$\text{Re}\beta \geq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}[(1 + p_i)M_i + 1]$$

for all $i \in \{1, ..., n\}$, then the integral operator $F_{\alpha_1,\alpha_2,...,\alpha_n,\beta}(z)$ defined by (2) is in the class $\mathcal{S}$.
Proof. We begin by setting

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}}dt.$$

By calculating the derivatives of the first and second order for the function $h(z)$, we obtain

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z} \right)^{\frac{1}{\alpha_i}}$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right). \tag{4}$$

We evaluate the modulus and we multiply in both terms of the relation (4) with $\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta}$, we obtain

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| &= \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right| \\ &\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left| \frac{zf_i'(z)}{f_i(z)} \right| + 1 \right) \\ &\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} \right| \left| \frac{f_i(z)}{z} \right| + 1 \right). \qquad (5)\end{aligned}$$

From the hypothesis of Theorem 2.1., we have

$$|f_i(z)| \leq M_i \qquad (z \in \mathbb{U}),$$

then by General Schwarz Lemma, we obtain that

$$|f_i(z)| \leq M_i \, |z| \qquad (z \in \mathbb{U})$$

for all $i \in \{1, 2, ..., n\}$.
Therefore, by using the inequalitities (1) and (5), we arrive at the following inequality:

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| &\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left(\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| + 1 \right) M_i + 1 \right) \\ &\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(p_i M_i \, |z|^2 + M_i + 1 \right) \\ &\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} [(1 + p_i) \, M_i + 1] \\ &\leq \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} [(1 + p_i) \, M_i + 1]\end{aligned}$$

since $\mathrm{Re}\beta \geq \sum_{i=1}^{n} \frac{1}{|\alpha_i|}[(1+p_i) M_i + 1]$. Applying Theorem 1.1., we conclude that the integral operator $F_{\alpha_1, \alpha_2, ..., \alpha_n, \beta}(z)$ defined by (2) is in class $\mathcal{S}$.

Setting $M_1 = M_2 = ... = M_n = M$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ in Theorem 2.1., we have

Corollary 2.2. Let the functions $f_i \in \mathcal{S}(p_i)$ satisfy the inequality (1), $0 < p_i \leq 2$ for all $i \in \{1, 2, ..., n\}$ and $M \geq 1$. Also, let α, β be complex numbers with the property $\mathrm{Re}\beta > 0$.
If

$$|f_i(z)| \leq M \qquad (z \in \mathbb{U})$$

and

$$\mathrm{Re}\beta \geq \sum_{i=1}^{n} \frac{1}{|\alpha|}[(1+p_i) M + 1]$$

for all $i \in \{1, ..., n\}$, then the integral operator

$$F_{\alpha,\beta}(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Setting n=1 in Theorem 2.1. we have

Corollary 2.3. Let the function $f \in \mathcal{S}(p)$ satisfies the inequality (1), $0 < p \leq 2$ and $M \geq 1$. Also, let α, β be complex numbers with the property $\mathrm{Re}\beta > 0$.
If

$$|f(z)| \leq M \qquad (z \in \mathbb{U})$$

and

$$\mathrm{Re}\beta \geq \frac{[(1+p) M + 1]}{|\alpha|}$$

then the integral operator

$$F(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t}\right)^{\frac{1}{\alpha}} dt\right)^{\frac{1}{\beta}}$$

is in the class $\mathcal{S}$.

Theorem 2.4. Let the functions $f_i \in S(p_i)$ satisfy the inequality (1), $0 < p_i \leq 2$ and $M_i \geq 1$ for all $i \in \{1, 2, ..., n\}$. Also, let α_i, β be complex numbers with $\beta = \left(\sum_{i=1}^{n} (\alpha_i - 1) + 1\right)$ and $\mathrm{Re}\beta > 0$. If

$$|f_i(z)| \leq M_i \qquad (z \in \mathbb{U})$$

and

$$\mathrm{Re}\beta \geq \sum_{i=1}^{n} |\alpha_i - 1| \left[(1+p_i) M_i + 1\right]$$

for all $i \in \{1, ..., n\}$, then the integral operator $G_{\alpha_1, \alpha_2, ..., \alpha_n}(z)$ defined by (3) is in the class $\mathcal{S}$.

Proof. Considering the same steps as in the proof of Theorem 2.1., we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} (\alpha_i - 1) \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right)$$

which readily shows that

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| = \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \sum_{i=1}^{n} (\alpha_i - 1) \left(\frac{z f_i'(z)}{f_i(z)} - 1 \right) \right|$$

$$\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} |\alpha_i - 1| \left(\left| \frac{z f_i'(z)}{f_i(z)} \right| + 1 \right)$$

$$\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} |\alpha_i - 1| \left(\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} \right| \left| \frac{f_i(z)}{z} \right| + 1 \right). \qquad (6)$$

From the hypothesis of Theorem 2.4., we have

$$|f_i(z)| \leq M_i \qquad (z \in \mathbb{U}),$$

then by General Schwarz Lemma, we obtain that

$$|f_i(z)| \leq M_i |z| \qquad (z \in \mathbb{U})$$

for all $i \in \{1, ..., n\}$.

Therefore, by using the inequalitities (1) and (6), we obtain

$$\frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} |\alpha_i - 1| \left(\left(\left| \frac{z^2 f_i'(z)}{[f_i(z)]^2} - 1 \right| + 1 \right) M_i + 1 \right)$$

$$\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} |\alpha_i - 1| \left(p_i M_i |z|^2 + M_i + 1 \right)$$

$$\leq \frac{1-|z|^{2\mathrm{Re}\beta}}{\mathrm{Re}\beta} \sum_{i=1}^{n} |\alpha_i - 1| \left[(1 + p_i) M_i + 1 \right]$$

$$\leq \frac{1}{\mathrm{Re}\beta} \sum_{i=1}^{n} |\alpha_i - 1| \left[(1 + p_i) M_i + 1 \right]$$

since $\mathrm{Re}\beta \geq \sum_{i=1}^{n} |\alpha_i - 1| \left[(1 + p_i) M_i + 1 \right]$. Applying Theorem 1.1., we conclude that the integral operator $G_{\alpha_1, \alpha_2, ..., \alpha_n}(z)$ defined by (3) is in the class $\mathcal{S}$.

Setting $M_1 = M_2 = ... = M_n = M$ and $\alpha_1 = \alpha_2 = ...\alpha_n = \alpha$ in Theorem 2.4., we have

Corollary 2.5. Let the functions $f_i \in \mathcal{S}(p_i)$ satisfy the inequality (1), $0 < p_i \leq 2$ for all $i \in \{1, ..., n\}$ and $M \geq 1$. Also, let α, β be complex numbers with $\beta = \left(\sum_{i=1}^{n} (\alpha - 1) + 1\right)$ and $\text{Re}\beta > 0$. If

$$|f_i(z)| \leq M \qquad (z \in \mathbb{U})$$

and

$$\text{Re}\beta \geq \sum_{i=1}^{n} |\alpha - 1| \left[(1 + p_i) M + 1\right]$$

for all $i \in \{1, ..., n\}$, then the integral operator

$$G_\alpha(z) = \left(\left(\sum_{i=1}^{n} (\alpha - 1) + 1\right) \int_0^z \prod_{i=1}^{n} (f_i(t))^{\alpha - 1} \, dt\right)^{\frac{1}{\left(\sum_{i=1}^{n} (\alpha-1)+1\right)}}$$

is in the class $\mathcal{S}$.

Setting n=1 in Theorem 2.4., we have

Corollary 2.6. Let the function $f \in \mathcal{S}(p)$ satisfies the inequality (1), $0 < p \leq 2$ and $M \geq 1$. Also, let α be a complex number with the property $\text{Re}\alpha > 0$. If

$$|f(z)| \leq M \qquad (z \in \mathbb{U})$$

and

$$\text{Re}\alpha \geq |\alpha - 1| [(1 + p) M + 1]$$

then the integral operator

$$G(z) = \left(\alpha \int_0^z (f(t))^{\alpha - 1} \, dt\right)^{\frac{1}{\alpha}}$$

is in the class $\mathcal{S}$.

Acknowledgement. This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007-2013 co-financed by the European Social Fund-Investing in People.

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