ON THE EXPONENTIAL DIOPHANTINE EQUATION $p^x + 1009^y = 2^z$

VÎRGOLICI, Horia
Department of Mathematics and Informatics
University Spiru Haret
hvirgolici@yahoo.com

Abstract
The aim of this paper is to give all nonnegative solutions (x, y, z) to the equation $p^x + 1009^y = 2^z$, where p is a positive rational prime number with $3 \leq p \leq 997$ (we discuss 167 equations).t

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1. Introduction

It is known that the equation

$$a^x + b^y = c^z \tag{1}$$

where a, b, c are prime numbers, has only finitely many solutions, but there is no algorithm to compute all the solutions (x, y, z). Some particular cases were treated: Nagell[7] found all solutions for $\max(a, b, c) = 7$, Makowski[4], Hadano[3], Uchiyama[9], Qi Sun, Xiaoming Zhou[8] and Xiaozhuo Yang[10] determined all solutions for $11 \leq \max(a, b, c) \leq 23$. Cao[2] found all solutions for $29 \leq \max(a, b, c) \leq 97$ (60 solutions in total).
The aim of this paper is to give all nonnegative solutions to equation (1) for $a = 2, c = 1009$ (1009 representing the first prime number > 1000) and b is a rational prime number, $3 \leq b \leq 997$. The main results are given by the following

Theorem 1 *The only equations $p^x + 1009^y = 2^z$, with p rational prime, $3 \leq p \leq 1009$, which admit nonnegative solutions (x,y,z) are (taking $a < b$):*

- $3^x + 1009^y = 2^z$, *which has the solution (1,0,2).*

- $7^x + 1009^y = 2^z$, *which has the solution (1,0,3).*
- $31^x + 1009^y = 2^z$, *which has the solution (1,0,5).*
- $127^x + 1009^y = 2^z$, *which has the solution (1,0,7).*

and all the equations $p^x + 1009^y = 2^z$ *have the trivial solution (x,y,z)=(0,0,1).*

The proof for Theorem 1 is given in subsection 3.

2. **Preliminaries**

Below we present a theorem which shows that the Catalan's equation has only one solution and gives this solution.

Theorem 2 *([1, 5, 6]). Equation (named Catalan's equation)*

$$a^x - b^y = 1 \tag{2}$$

has no solutions in integers $a,\ b,\ x,\ y > 1$ *other than* $3^2 - 2^3 = 1$.

3. **Proofs of the main results**

We give the proof for Theorem 1, which treats the equation

$$p^x + 1009^y = 2^z,\ 3 \leq p \leq 997, p \text{ rational prime} \tag{3}$$

All these equations admit the trivial solution (x,y,z)=(0,0,1). If x=0, the equation (3) has no solutions (except the trivial solution), due to Theorem 2. Thus we find the solutions with x > 0.

Many of equations (3) have no solutions (x, y, z), except the trivial solution (0,0,1), due to:

Lemma 1 *If p ≡ 3 (mod 8), except a=3 (which is treated separately), or p ≡ 1,5,7,11,13,17,19,23,25,29,37,41,43,47 (mod 48),except p=7 (which is treated separately), then the equation (3) has no nonnegative nontrivial solutions (x,y,z).*

Proof. It is obvious, taking the equation (3) mod 8 and mod 48. □

Thus, we discuss the following equations, in which a doesn't comply with the conditions in the Lemma 1:

i/ $p^x + 1009^y = 2^z$, where $p \in \{271, 367, 607\}$; by taking the equation mod 252, it results that the equation has no solutions.

ii/ $3^x + 1009^y = 2^z$; by taking the equation mod 3, it results that z is even; by taking the equations mod 8, it results that the equation has no solutions, except (1,0,2).

iii/ $7^x + 1009^y = 2^z$; by taking the equation mod 3, it results that z is odd; by taking the equations mod 16, it results that the equation has no solutions, except (1,0,3).

iv/ $31^x + 1009^y = 2^z$; by taking the equation mod 24, it results that x and z are odd; by taking the equations mod 5, it results that y is even and z $\equiv 1 \pmod 4$; by taking the equation mod 7, it results that x $\equiv 1 \pmod 6$ and z $\equiv 2 \pmod 3$; it results from these relations that z $\equiv 5 \pmod{12}$; by taking the equation mod 13, it results that x $\equiv 1 \pmod 4$ and y $\equiv 0 \pmod 4$; then it results that x $\equiv 1 \pmod{12}$; if y=0, the equation has the solution (1,0,5) and this is the only solution in this case, because for x > 1 the equation $31^x + 1 = 2^z$ has no solutions, due to Theorem 2; if y$>$ 0, taking into account that x $\equiv 1 \pmod{12}$ and z$\equiv 5 \pmod{12}$ and by taking the equation mod 1009, it results that the equation has no solutions.

v/ $79^x + 1009^y = 2^z$; by taking the equation mod 24, it results that x and z are odd; by taking the equations mod 5, it results that y is odd and z $\equiv 3 \pmod 4$; by taking the equation mod 13 and taking into account the relations obtained till now, it results that the equation has no solutions.

vi/ $127^x + 1009^y = 2^z$; by taking the equation mod 24, it results that x and z are odd; by taking the equation mod 7, it results that z $\equiv 1 \pmod 3$; it results from these relations that z $\equiv 1 \pmod 6$; by taking the equation mod 32, it results that y is even; by taking the equations mod 5, it results that x $\equiv 1 \pmod 4$ and z $\equiv 3 \pmod 4$; then it results that z $\equiv 7 \pmod{12}$; by taking the equation mod 13, it results that x $\equiv 1 \pmod 6$ and y $\equiv 0 \pmod 4$ or x $\equiv 3 \pmod 6$ and y $\equiv 2 \pmod 4$; from these relations it results that x $\equiv 1 \pmod{12}$ or x $\equiv 9 \pmod{12}$; if y=0, the equation has the solution (1,0,7) and it is the only solution in this case, because for x > 1 the equation $127^x + 1 = 2^z$ has no solutions, due to Theorem 2; if y$>$ 0, taking into account that x $\equiv 1 \pmod{12}$ or x $\equiv 9 \pmod{12}$ and z$\equiv 7 \pmod{12}$ and by taking the equation mod 1009, it results that the equation has no solutions.

vii/ $223^x + 1009^y = 2^z$, which has no solutions, by taking the equation mod 56.

viii/ $463^x + 1009^y = 2^z$; by taking the equation mod 8, it results that x is odd; by taking the equation mod 7, it results that z $\equiv 1 \pmod 3$; by taking the equation mod 9, it results that x $\equiv 1 \pmod 3$; from these relations, it results that x $\equiv 3 \pmod 6$ and z $\equiv 1 \pmod 6$; taking into account these relations and by taking the equation mod 13, it results that it has no solutions.

ix/ $751^x + 1009^y = 2^z$; by taking the equation mod 24, it results that x and z are odd; by taking the equation mod 5, it results that y is even and z $\equiv$ 1 (mod 4); by taking the equation mod 7, it results that x $\equiv$ 0 (mod 3) and z $\equiv$ 1 (mod 3); it results from the relations above that z $\equiv$ 1 (mod 12) and x $\equiv$ 3 (mod 6); taking into consideration these relations and by taking the equation mod 13, it results that the equation has no solutions.

x/ $991^x + 1009^y = 2^z$; by taking the equation mod 24, it results that x and z are odd; by taking the equation mod 5, it results that y is even and z $\equiv$ 1 (mod 4); by taking the equation mod 7, it results that x $\equiv$ 0 (mod 3) and z $\equiv$ 1 (mod 3); it results from the relations above that z $\equiv$ 1 (mod 12) and x $\equiv$ 3 (mod 6); by taking the equation mod 13 and taking into consideration that z $\equiv$ 1 (mod 12), it results that y $\equiv$ 0 (mod 4); taking into account these relations and by taking the equation mod 64, it results that the equation has no solutions.

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