DETERMINATION OF THE NUMBER OF ISOMORPHIC GRAPHS WITH DIAGONAL CELL TYPE WALKS MATRIX

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Abstract
Considering a graph G my goal is to determine the number of isomorphic graph H with G for which H has a diagonal cell type walks matrix. For this operation I wish to use only combinatorics and graph theory (without using isomorphism properties from algebra).
Firstly I present some of graph theory definition used in this paper. I divide my presentation in two simple cases (equal dimensioned graph components and distinct dimensioned graph components) and then I consider the general case.

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1. Introduction

Firstly I wish to present some of graph theory definitions. I make this because there are authors that used different definitions, starting with definition for graph. In such a definition, a graph G is defined as (V, E, f) where V represent the vertices, E is the set of edges and f is adjacency function.

The definition used by others (including myself) is the following:

Definition 1.1 *A graph G is a pair (V, E) where V is a finite set specifying the vertices (generally considered as $V = \{1, 2, \ldots, n\}$) and E is the set of unordered pairs of numbers from V (generally presented as subsets of two values from V) named edges.*

In the above definition we do not assume that numbers from V are different so we can not have $\{x, x\} \in E$. More then that, we do not assume that if a and b are two different pairs from E, then we can not have a and b formed with the same values x and y from V.

Definition 1.2 *Let $G = (V, E)$ be a graph and x and y two different vertices from G. We have a walk from x to y if and only if we have a sequence*

$x = v_1, v_2, \ldots, v_k = y$ of vertices so that, for any $i = 1, 2, \ldots, k-1$, $\{v_i, v_{i+1}\}$ is an edge in E. If $x = y$, the walk is named loop. We specify the walk from x to y by $[x = v_1, v_2, \ldots, v_k = y]$.

Definition 1.3 *Let $G = (V.E)$ be a graph. G is a connected graph if and only if, for any different vertices x and y there exists a walk from x to y. If G is not connected then G is named disconnected graph.*

Definition 1.4 *Let $G = (V, E)$ be a graph and W is a subset of V. We define the subset F of E as $F = \{u = \{x, y\} \in E | x, y \in W\}$ and a new graph $H = (W, F)$. H is named subgraph of G.*

Definition 1.5 *Let $G = (V, E)$ be a graph. A subgraph $H = (W, F)$ of G is named component of G if and only if H is a connected graph and for any $x \in W$ and any $y \in V - W$ do not exist an edge $\{x, y\} \in E$.*

If G is a connected graph then the only component of G is G itself.

If G is a disconnected graph one can demonstrate that there exists a partition of V, $W_1, W_2, \ldots, W_k$, so that, defining $F_1, F_2, \ldots, F_k$ as above for any $i = 1, 2, \ldots, k$, the subgraph $H_i = (W_i, F_i)$ is a component of G.

There are some matrixes that can be associated with graphs. One of this matrix is adjacencies matrix, $A_G = (a_{ij})_{i,j=1,2,\ldots,n}$, where $a_{ij} = 1$ if and only if there exist a walk from i to j, otherwise $a_{ij} = 0$, and n represents the number of vertices. For this paper the next definition is important:

Definition 1.6 *Let $G = (V, E)$ be a graph having n vertices. We define walks matrix as a $0, 1$ matrix $D_G = (d_{ij})_{i,j=1,2,\ldots,n}$, where $d_{ij} = 1$ if and only if there exist a walk from i to j , otherwise $d_{ij} = 0$.*

There are algorithms to compute walks matrix D_G from adjacencies matrix A_G.

Another important definition for this paper indicate a special form for 0,1 matrix. This definition is:

Definition 1.7 *Let A be a $0, 1$ n-dimensioned matrix so that there exist numbers $n_1, n_2, \ldots, n_k$, not necessarily different, with $n_1 + n_2 + \ldots + n_k = n$ so that*

$$A = \begin{pmatrix} B_1 & O & \ldots & O \\ O & B_2 & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & B_K \end{pmatrix},$$

where for any $i = 1, 2, \ldots, k$ B_i is a n_i-dimensioned matrix with all elements equal 1 and O represent matrix with all elements equal 0. Matrix A of this form is named diagonal cell type matrix.

My aim is to solve the following problem using only combinatorics.

Problem *Giving a graph $G = (V, E)$, let determine the number of graphs $H = (V, F)$ isomorphs with G so that H has a diagonal cell type walks matrix.*

Here I must specify that this problem is already solved in terms of algebras.

2. **Isomorphic graphs**

In this section I present some results related to isomorphic graphs which is necessary to prepare my calculus. Firstly we must know what it means isomorphic graphs.

Definition 2.1 *Let $G = (V, E)$ and $H = (W, F)$ be two graphs. G and H are named isomorphic if and only if there exists a function $f : V \to W$ so that f is bijective and $a = \{x, y\} \in E$ if and only if $f(a) = \{f(x), f(y)\} \in F$.*

One can demonstrate that:

Note 2.1 *If $G = (V, E)$ and $H = (V, F)$ are two isomorphic graphs and σ is a permutation of V so that $F = \{\{\sigma(x), \sigma(y)\} \mid \{x, y\} \in E\}$, we write $H = \sigma(G)$.*

If $G = (V, E)$ is a connected graph with n vertices then $D_G = B_n$ is a matrix with all elements equal 1. If $G = (V, E)$ is a connected graph and σ is a permutation of V, then $H = \sigma(G)$ is a connected graph and so $D_H = B_n$. So we have that for any permutation σ of V, $\sigma(G)$ isomorphic with G and $D_{\sigma(G)} = B_n$. It results that the number of isomorphic graph H with G for which $D_H = B_n$ is equal with the number of permutation of V and so it is $n!$. We can consider that B_n is diagonal cell type matrix (having only one cell) and so we have shown the following result:

Theorem 2.2 *For any graph $G = (V, E)$ there exists a permutation σ of V so that $H = \sigma(G)$ has diagonal cell type walks matrix.*

If $G = (V, E)$ is a connected graph with n vertices then $D_G = B_n$ is a matrix with all elements equal 1. If $G = (V, E)$ is a connected graph and σ is a permutation of V, then $H = \sigma(G)$ is a connected graph and so $D_H = B_n$. So we have that for any permutation σ of V, $\sigma(G)$ isomorphic with G and $D_{\sigma(G)} = B_n$. It results that the number of isomorphic graph H with G for which $D_H = B_n$ is equal with the number of permutation of V and so it is $n!$. We can consider that B_n is diagonal cell type matrix (having only one cell) and so we have shown the following result:

Lemma 2.3 *Giving a connected graph $G = (V, E)$, the number of graphs $H = (V, F)$ isomorphic with G so that H has a diagonal cell type walks matrix is $n!$.*

This result represents a solution for our problem when G is a connected graph. So our problem is to be solved for disconnected graphs.

3. **Particular cases**

In this section I solve my problem for two particular cases. In first case the graph G is disconnected and all components of G have the same dimension (number of vertices). In second case the graph G is disconnected and components of G have distinct dimension. In both cases I'll show that, if G has only one component, then the result from above lemma still works.

3.1. *Equal dimension components graph*

Let consider a disconnected graph $G = (V, E)$ with n vertices. Then there exists a partition of V with subsets $V_1, V_2, \dots, V_k$ so that if

$$E_i = \{\{x, y\} \in E | x, y \in V_i\}$$

then $G_i = (V_i, E_i)$ is component in G. We assume that, for any $i = 1, 2, \dots, k$, V_i has p vertices. Because $V_1, V_2, \dots, V_k$ represent a partition of V it result that $n = kp$.

From the theorem presented in section 2, for G disconnected result that there exists a permutation σ of V so that $\sigma(G) = H$ has a diagonal cell type walks matrix.

We consider $W_i = \{\sigma(y) | y \in V_i\}$ for any $i = 1, 2, \dots, k$ and because $V_1, V_2, , V_k$ is a partition of V it follows that $W_1, W_2, \dots, W_k$ is also a partition of V. Because $G_i = (V_i, E_i)$ is connected it results that $H_i = (W_i, F_i)$ is connected, where

$$F_i = \left\{\{x, y\} \,|\, \left\{\sigma^{-1}(x), \sigma^{-1}(y)\right\} \in E_i\right\}.$$

If we assume that for $x \in W_i$, $y \in W_j$, there exists $\{x, y\}$ in graph $\sigma(G) = H$, it results that $\{\sigma^{-1}(x), \sigma^{-1}(y)\}$ is an edge in G and so $i = j$. So $H_1, H_2, \dots, H_k$ are components in H.

Because all components in G have the same number p of vertices, it results that all components in H have the same number p of vertices. Because H has diagonal cell type walks matrix, it follows that any cell is p-dimensioned and diagonal cell type walks matrix for H is:

$$D_H = \begin{pmatrix} B_p & O & \dots & O \\ O & B_p & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & B_p \end{pmatrix},$$

where B_p is a p-dimensioned matrix with all elements equal 1 and O represent matrix with all elements equal 0.

In addition, values in H_i are consecutive integers.

We fix a component i and we consider a permutation σ_i of V so that restriction of σ_i to $V - W_i$ is identity permutation, and so σ_i change values only inside of W_i.

Because σ_i is a permutation of V, $\sigma_i(H)$ is isomorphic with H (and also with G). Because restriction of σ_i to $V - W_i$ is identity, it follows that in $\sigma_i(H)$ al components are unchanged, except the component corresponding to W_i. It results that the number of isomorphic graphs with H generated by permutation of type σ_i is equal with the number of permutation of W_i and so is equal with $p!$.

Because we fix independently the component i, if we generate a permutation τ of V for which $\tau(H)$ preserves the components $W_1, W_2, \dots, W_k$ in this order, $\tau(H)$ is isomorphic with H and $\tau = \sigma_1\sigma_2\ldots\sigma_k$. Because of independency in fixing component i, the number of such permutation τ is equal with $(p!)^k$.

Above we considered a given order for component of H, namely

$$W_1, W_2, \ldots, W_k.$$

If we do not have this constraint, we must consider all possible order for components, and so we have $k!$.

We showed the following result:

Theorem 3.1 *If $G = (V, E)$ is a graph with k components and all components have the same number p of vertices, then the number of isomorphic graph H with G for which H has a diagonal cell type walks matrix is $k!(p!)^k$.*

To complete this result we must observe that if G is a connected graph, then the number of component is $k = 1$ and number of vertices is $p = n$ and so we have $1!(n!)^1 = n!$ isomorphic graph H with G and H has a diagonal cell type walks matrix, result that is identical with that specified at the end of section 2.

3.2. *Distinct dimension components graph*

Let consider a disconnected graph $G = (V, E)$ with n vertices. Then there exists a partition of V with subsets $V_1, V_2, \ldots, V_k$ so that if

$$E_i = \{\{x, y\} \in E | x, y \in V_i\}$$

then $G_i = (V_i, E_i)$ is component in G for any $i = 1, 2, \ldots, k$. We assume that for any $i = 1, 2, \ldots, k$, V_i has p_i vertices and $p_i \neq p_j$ for any $1 \leq i, j \leq k$, $i \neq j$. Because $V_1, V_2, \ldots, V_k$ is a partition of V it follows that $n = \sum_{i=1}^{k} p_i$. From this assumption it results that the set $\{p_1, p_2, \ldots, p_k\}$ has k elements.

Using the theorem from section 2 for disconnected graph G we have that there exists a permutation σ of V so that $\sigma(G) = H$ has a diagonal cell type walks matrix.

As in subsection 3.1, let consider $W_i = \{\sigma(y) | y \in V_i\}$ and

$$F_i = \left\{ \{x, y\} \,|\, \left\{ \sigma^{-1}(x), \sigma^{-1}(y) \right\} \in E_i \right\}$$

for any $i = 1, 2, \ldots, k$. Because $V_1, V_2, \ldots, V_k$ is a partition of V it follows that $W_1, W_2, \ldots, W_k$ is a partition of V. Because $G_i = (V_i, E_i)$ is connected it follows that $H_i = (W_i, F_i)$ is also connected. More then that, $H_1, H_2, \ldots, H_k$ are the components of H.

If we consider that r_i is the number of vertices in H_i, because $p_i \neq p_j$ for any $1 \leq i, j \leq k$, $i \neq j$, it follows that $r_i \neq r_j$ for any $1 \leq i, j \leq k$, $i \neq j$ and so $\{p_1, p_2, \ldots, p_k\} = \{r_1, r_2, \ldots, r_k\}$.

Because H has diagonal cell type walks matrix and any cell correspond to a component in H it follows that the diagonal cell type walks matrix for H

is:

$$D_H = \begin{pmatrix} B_{r_1} & O & \dots & O \\ O & B_{r_2} & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & B_{r_k} \end{pmatrix},$$

where B_{r_i} is a r_i-dimensioned matrix with all elements equal 1.

We fix a component i and we consider a permutation σ_i of V so that $\sigma_i(j) = j$ if $j \in V - W_i$ and so σ_i changes values only inside of W_i. It follows that $\sigma_i(H)$ is isomorphic with H and in $\sigma_i(H)$ all components are identical excepting those that correspond to W_i. It results that number of isomorphic graph with H generated by permutation of type σ_i, for i fixed, is equal with the number of permutation of W_i and so is equal with $r_i!$.

Because we fix the component i independently, it means that for any $i = 1, 2, \dots, k$ we have $r_i!$ possibilities.

Now we can generate a permutation τ of V for which $\tau(H)$ preserve the components $W_1, W_2, \dots, W_k$ in this order, $\tau(H)$ is isomorphic with H and $\tau = \sigma_1\sigma_2 \dots \sigma_k$. Because $\{p_1, p_2, \dots, p_k\} = \{r_1, r_2, \dots, r_k\}$ and independency in fixing component i, the number permutation τ is equal with

$$\prod_{i=1}^{k} r_i! = \prod_{i=1}^{k} p_i!.$$

We considered a given order for component of H, namely $W_1, W_2, \dots, W_k$. If we do not have this constraint, we must consider all possible order for components, and so we have $k!$ possible orders.

We showed the following result:

Theorem 3.2 *If $G = (V, E)$ is a graph with k components, each component with p_i vertices and for any $1 \leq i, j \leq k$, $i \neq j$, $p_i \neq p_j$, then the number of isomorphic graph H with G for which H has a diagonal cell type walks matrix is:*

$$k! \prod_{i=1}^{k} p_i!.$$

To complete this result we must observe that if G is a connected graph, then the number of components is $k = 1$ and the number of vertices is $p_1 = n$. We have $1!p_1! = n!$ isomorphic graph H with G and H has a diagonal cell type walks matrix and so the result presented at the end of section 2 remains valid.

The results presented in section 3 represent solutions for two particular cases for given problem. To solve completely the problem we must consider the general case of a graph G and to demonstrate that what we obtain remains valid in particular cases from above.

4. **General case**

Let $G = (V, E)$ be a graph with n vertices. Without losing generality we can consider that G is disconnected. If G is connected, from the result presented at the end of section 2 we have $n!$ isomorphic graph H with G and H has a diagonal cell type walks matrix, which is the solution to our problem.

For G disconnected there exists a partition $V_1, V_2, \ldots V_k$ of V so that if

$$E_i = \{\{x, y\} \in V | x, y \in V_i\}$$

and $G_i = (V_i, E_i)$, then $G_1, G_2, \ldots, G_k$ are components of G.

We can assume from the beginning that G has a diagonal cell type walks matrix. If it is not so, there exists a permutation σ of V so that $\sigma(G) = H$ is isomorphic with G and H has a diagonal cell type walks matrix and we can consider H instead of G.

Let $\{p_1, p_2, \ldots, p_r\}$ be the set of numbers of vertices in $V_1, V_2, \ldots, V_k$, where $1 \leq r \leq k$. If $r = 1$ then all sets $V_1, V_2, \ldots, V_k$ have the same number of vertices and this is the case analyzed in subsection 3.1. If $r = k$ then any set V_i has p_i vertices and it follows that $p_i \neq p_j$ for $1 \leq i, j, \leq k$, $i \neq j$ which is the case analyzed in subsection 3.2. So without losing generality we can assume that $1 < r < k$. In this case it results that there exists $i \neq j$, $1 \leq i, j \leq k$ so that V_i and V_j have the same number of vertices. Now let consider that for any i, $1 \leq i \leq r$, c_i components in G have the number of vertices equal with p_i.

From the above consideration it results that

$$\sum_{i=1}^{k} c_i = k$$

and

$$\sum_{i=1}^{r} c_i p_i = n.$$

In the same way in which we worked in subsections 3.1 and 3.2 we can fix a component i, $1 \leq i \leq k$, and consider a permutation σ_i of V so that $\sigma_i(j) = j$ if $j \in V - V_i$. Because σ_i changes values only inside V_i it follows that the number of permutation of type σ_i is equal with the number of permutation of V_i, so is $|V_i|!$.

If we extend the fixing on components with p_j vertices, $1 \leq j \leq r$, namely $V_{s_1}, V_{s_2}, \ldots, V_{s_{c_j}}$ then we can define a permutation τ_{p_j} of V so that $\tau_{p_j} = \sigma_1 \sigma_2 \ldots \sigma_{c_j}$. This permutation changes values only inside the sets $V_{s_1}, V_{s_2}, \ldots, V_{s_{c_j}}$ and because $\sigma_1, \sigma_2, \ldots, \sigma_{c_j}$ are chosen independently it follows that the number of permutation of type τ_{p_j} is equal with $(p_j!)^{c_j}$.

Using the results from above we can now define a permutation ϵ of V so that $\epsilon = \tau_{p_1} \tau_{p_2} \ldots \tau_{p_r}$. Because we fix independently the components with the same number p_j of vertices it follows that the number of permutation of type ϵ is equal with

$$\prod_{j=1}^{r} (p_j!)^{c_j}.$$

To obtain this result we consider a given components order for G, namely the original order $V_1, V_2, \ldots, V_k$. If this constraint doesnt exist we can generate a convenient permutation α so that $\alpha(G) = H$ is isomorphic with G, H has a diagonal cell type walks matrix, but the order of components in H is different from original order $V_1, V_2, \ldots, V_k$.

To calculate the number of permutation of type α we must observe that, firstly we can fix the positions for all components with the number of vertices equal with p_1. We have to fix c_1 component in k places considering every possible order and so we have

$$A_k^{c_1} = \frac{k!}{(k - c_1)!}$$

possibilities and it remains to cover $k - c_1$ positions.

Secondly we fix all components with p_2 vertices. We have to fix c_2 components in $k - c_1$ places and so we have

$$A_{k-c_1}^{c_2} = \frac{(k - c_1)!}{(k - c_1 - c_2)!}$$

possibilities and so on.

Finally we have the number of possible fixing equal with

$$\frac{k!}{(k - c_1)!} \frac{(k - c_1)!}{(k - c_1 - c_2)!} \cdots \frac{(k - \sum_{i=1}^{j-1} c_i)!}{(k - \sum_{i=1}^{j} c_i)!} \cdots \frac{(k - \sum_{i=1}^{r-1} c_i)!}{(k - \sum_{i=1}^{r} c_i)!} = \frac{k!}{0!} = k!$$

possibilities for permutation of type α.

So we showed the following result which gives the solution for our problem:

Theorem 4.1 *If $G = (V, E)$ is a graph with k component so that the set of number of vertices is $\{p_1, p_2, \ldots, p_r\}$ and the number of components with p_i vertices is c_i for any $i = 1, 2, \ldots, r$, then the number of graph H isomorphic with G and H has a diagonal cell type walks matrix is*

$$k! \prod_{j=1}^{r} (p_j!)^{c_j}.$$

We can observe that for $r = 1$ we have the result specified in theorem 3.1 and for $r = k$ we have the result indicated in theorem 3.2.

5. Conclusion and remarks

The subject of this paper is not a new one. What I have tried to do is to offer a possible solving using only combinatorics tools.

The problem is already solved using algebras tools. Such a solution is given in *Combinatorica si teoria grafurilor* (Combinatorics and Graph theory) ([4]).

Proposed solution could indicate a practical way to generate all isomorphic graphs with a given one and for which walks matrix is a diagonal cell type matrix.

A consequence of the results presented in this paper can be the determination of the number of isomorphic graph with a given graph with diagonal cell type walks matrix for which the walks matrix is identical. This problem is solved in another paper already sent for publishing.

To continue this subject I wish to study the corresponding problem for digraphs with using paths matrix.

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Finally I want to remark that all preliminary definitions and results are english version for those in ([2])

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