

PAIRS OF GLAUERT INTEGRALS

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Abstract

Based on the requirement of the analytical computation of some singular integrals involved in the theoretical aerodynamics, certain considerations about the twin Glauert's integral have been made. New techniques and some computation rules have been established.

Keywords: *analytic integration, asymptotic expansion, cosine logarithm*

AMS Classification: 65D30, 74H10, 74H15

1. Introduction

This paper aims analytical calculation of integrals of Glauert type. Starting mainly from Glauert's work [5], many authors have had to deal with this topic. Approaching different techniques, they have many ways to calculate the Glauert's integral [3], [7]. It should be noted that Glauert introduces one of the most important issues that occur in the singular integral theory of hydrodynamic or thin wings aerodynamics, as we can see in various works like [3], [9], [8] and many other. We have made also use of such singular integrals at the incompressible flows past oscillating airfoils [1], [2], [6].

The purpose of this work is to find new computation techniques for the pairs of Glauert's integrals. Certain practical rules will be also deduced from our exposure.

2. Computation techniques of twin Glauert's integrals

All over this paper we will refer to what we will mean by the twin Glauert type integrals

$$\mathcal{G}_n(\sigma) = \frac{1}{\pi} \int_0^{\pi} \frac{\cos n\theta}{\cos \theta - \cos \sigma} d\theta, \quad \mathcal{H}_n(\sigma) = \frac{1}{\pi} \int_0^{\pi} \frac{\sin n\theta}{\cos \theta - \cos \sigma} d\theta \quad (1)$$

and their consequences used in computation scheme of theoretical aerodynamics issues. First of the integral was calculated by many authors see for instance [3], [7]. We here present another method based on a simple asymptotic expansion of a difference cosine logarithm [3]. Since $\mathcal{G}_n = \mathcal{G}_{-n}$, $\mathcal{H}_{-n} = -\mathcal{H}_n$ in the

next we may presume that $n \in \mathbf{N}$.

In order to find exact formulas for the above integrals, we will introduce another pair of improper integrals associated to those defined by (1),

$$\mathcal{C}_n(\sigma) = \int_0^{\prime\pi} \cos n\theta \ln |\cos \theta - \cos \sigma| d\theta, \quad \mathcal{S}_n(\sigma) = \int_0^{\prime\pi} \sin n\theta \ln |\cos \theta - \cos \sigma| d\theta \quad (2)$$

for which we may establish below the first of the following our results.

For convenience, we shall consider that a parameter series may be written as

$$\sum_{m \geq 1} a_{m,n} = \sum_{m \neq n} a_{m,n} + \sum_{m = n} a_{m,n} \quad (3)$$

where the first of the sum is considered for all terms $a_{m,n}$, $m \neq n$ provided that $a_{n,n} = 0$, and the second one, for all terms $a_{n,n}$ when $a_{m,n} = 0$, except for the usual case of series

$$\sum_{m \geq 1} b_m$$

Proposition 1. ($\mathcal{S}_n(\sigma), \mathcal{C}_n(\sigma)$ computation formulas). For any $n \geq 1$, it takes place the computation formulas

$$\mathcal{C}_n(\sigma) = -\frac{\pi \cos n\sigma}{n}, \quad \mathcal{S}_n(\sigma) = \frac{\ln 2}{n} [(-1)^n - 1] + 2n \sum_{m \neq n} \frac{(-1)^{m+n} - 1}{m(m^2 - n^2)} \cos m\sigma \quad (4)$$

Particularly, for $n = 0$ we have $\mathcal{C}_0(\sigma) = -\ln 2$, $\mathcal{S}_0(\sigma) = 0$.

Proof. Taking into consideration the fundamental formula [6]

$$\ln(2 |\cos \theta - \cos \sigma|) = -2 \sum_{m \geq 1} \frac{1}{m} \cos m\sigma \cos m\theta \quad (5)$$

the right member of (2)₁ may be written as

$$\mathcal{C}_n(\sigma) = -2 \sum_{m \neq n} \frac{\cos m\sigma}{m} A_{m,n}$$

where the coefficients $A_{m,n}$ stand for [6]

$$A_{m,n} = \int_0^{\prime\pi} \cos m\theta \cos n\theta d\theta = \begin{cases} \frac{\pi}{2}, & m = n \\ 0, & m \neq n \end{cases} \quad (6)$$

From the last two relations we get the formula (4)₁. Particularly, for $n = 0$, from the last two relations, we get $\mathcal{C}_0(\sigma) = 0$. Furthermore, after an integration into the right side of relation (2)₂, we may write

$$\mathcal{S}_n(\sigma) = \frac{\ln 2}{n} [(-1)^n - 1] - 2 \sum_{m \neq n} B_{m,n} \frac{\cos m\sigma}{m} \quad (7)$$

where by the coefficients $B_{m,n}$ we mean [6]

$$B_{m,n} = \int_0^{\pi} \cos m\theta \sin n\theta d\theta = \begin{cases} \frac{n}{m^2 - n^2} [1 - (-1)^{m+n}] & m \neq n \\ 0 & m = n \end{cases} \quad (8)$$

Replacing (8) into (7) one leads to the formula (2)₂. Taking $n = 0$ in (2)₂, we find $\mathcal{S}_0(\sigma) = 0$.

Consequence 1. Particularly, by differentiating into (4) with respect to σ , we get

$$\mathcal{C}'_n(\sigma) = \pi \sin n\sigma, \quad \mathcal{S}'_n(\sigma) = 2n \sum_m^{\neq} \frac{1 - (-1)^{m+n}}{m^2 - n^2} \sin m\sigma \quad (9)$$

Proposition 2 (*Evaluation of Glauert integrals*). For any $n \geq 0$ it takes place the formula

$$\mathcal{G}_n(\sigma) = \pi \frac{\sin n\sigma}{\sin \sigma}, \quad \mathcal{H}_n(\sigma) = \frac{2n}{\sin \sigma} \sum_m^{\neq} \frac{1 - (-1)^{m+n}}{m^2 - n^2} \sin m\sigma \quad (10)$$

Particularly, for $n = 0$ we have $\mathcal{G}_0(\sigma) = \mathcal{H}_0(\sigma) = 0$.

Proof. Applying the differentiation of the parameter integral into (2), we may simply write

$$\mathcal{C}'_n(\sigma) = \pi \sin \sigma \mathcal{G}_n(\sigma), \quad \mathcal{S}'_n(\sigma) = \sin \sigma \mathcal{H}_n(\sigma) \quad (11)$$

and further, by comparing the relations (??) and (11), the formulas (10) will then be found.

Remark 1

Next, we consider three of improper integral which play an important role in the theoretical aerodynamics computation

$$I_{m,n} = \int_0^{\pi} \frac{\sin m\theta \sin n\theta}{\cos \theta - \cos \sigma} d\theta, \quad J_{m,n} = \int_0^{\pi} \frac{\cos m\theta \cos n\theta}{\cos \theta - \cos \sigma} d\theta \quad (12)$$

$$K_{m,n} = \int_0^{\pi} \frac{\sin m\theta \cos n\theta}{\cos \theta - \cos \sigma} d\theta \quad (13)$$

Consequence 2. Using the formula (10) we get the following results

$$I_{m,n} = -\pi \frac{\cos n\sigma \sin m\sigma}{\sin \sigma}, \quad J_{m,n} = \pi \frac{\sin n\sigma \cos m\sigma}{\sin \sigma} \quad (14)$$

Moreover, if $n \geq m \geq 0$ we may write

$$K_{m,n} = \frac{1}{2} (\mathcal{H}_{n+m}(\sigma) - \mathcal{H}_{n-m}(\sigma)) \quad (15)$$

Remark 2. Particularly, from (15) we get $K_{n,0} = H_n$.

Proof. By changing the product into a sum or difference of cosines inside the integral we have

$$I_{m,n} = \frac{1}{2}(\mathcal{G}_{m-n}(\sigma) - \mathcal{G}_{m+n}(\sigma)), \quad J_{m,n} = \frac{1}{2}(\mathcal{G}_{m-n}(\sigma) + \mathcal{G}_{m+n}(\sigma))$$

and after some simplifications, we get the formulas (14). For $m = 1$ the computation formula may be expressed by a cosines series as it follows.

Proposition 3 (*Useful formula*). For any $n \geq m \geq 0$ it takes places estimation

$$K_{1,n} = 2 \sum_m^{\neq} \frac{\cos m\sigma}{m} [(-1)^{m+n} - 1] \left(1 - \frac{n^2}{m^2 - n^2}\right) \quad (16)$$

Proof. Indeed, if we integrate in the right side of (2)₂ then, it may also be written as

$$n\mathcal{S}_n(\sigma) = \ln |1 - \cos \sigma| - (-1)^n \ln |1 + \cos \sigma| - K_{1,n}(\sigma)$$

where using the fundamental formula (5) the difference of logarithms we may replace by

$$\ln |1 - \cos \sigma| - (-1)^n \ln |1 + \cos \sigma| = [(-1)^{m+n} - 1] \ln 2 + 2 \sum_m^{\neq} \frac{\cos m\sigma}{m} [(-1)^{m+n} - 1]$$

and further, taking into account (2)₂, the sought relation (16) will be found.

Consequence 3 (*Recurrence formulas*). For any $n \geq 1$ it takes place the recurrence formulas

$$\frac{1}{2}[\mathcal{G}_{n+1}(\sigma) + \mathcal{G}_{n-1}(\sigma)] = \frac{(-1)^n - 1}{n} + \cos \sigma \mathcal{G}_n(\sigma) \quad (17)$$

$$\frac{1}{2}[\mathcal{H}_{n+1}(\sigma) + \mathcal{H}_{n-1}(\sigma)] = \frac{1 - (-1)^n}{n} + \cos \sigma \mathcal{H}_n(\sigma) \quad (18)$$

Proof. In order to prove the first two relations we may take into account the relations

$$\cos(n+1)\theta + \cos(n-1)\theta = 2 \cos n\theta (\cos \theta - \cos \sigma) + 2 \cos \sigma \cos n\theta$$

$$\sin(n+1)\theta + \sin(n-1)\theta = 2 \sin n\theta (\cos \theta - \cos \sigma) + 2 \cos \sigma \sin n\theta$$

after that a direct integration leads us to (17)-(18). The last relation may be found if we change the sum into a sines product.

3. Applications

First of all, let us see that for $n = 1$ and $n = 2$, the formulas which are obtained if it leaves from the definition $(1)_2$ will yield same results as if when we apply the computation relation $(10)_2$. Indeed, considering $n = 1$ in $(1)_2$ then, by a direct integration we get

$$\mathcal{H}_1(\sigma) = \int_0^{\pi} \frac{\sin \theta}{\cos \theta - \cos \sigma} d\theta = \ln \left| \frac{1 - \cos \sigma}{1 + \cos \sigma} \right| \quad (19)$$

But if one takes into account the fundamental formula (5), the right side in (19) can be replaced by

$$\ln \left| \frac{1 - \cos \sigma}{1 + \cos \sigma} \right| = 2 \sum_{m=1}^{+\infty} \frac{\cos m\sigma}{m} [(-1)^m - 1] = K_{1,0}(\sigma) \quad (20)$$

On the other hand, leaving from the definition $(1)_2$, we may find the equivalent of representation

$$\mathcal{H}_1(\sigma) = \frac{2}{\sin \sigma} \sum_m^{\neq} \frac{1 - (-1)^m}{m^2 - 1} \sin m\sigma \quad (21)$$

For $n = 2$ one may apply the recurrence formula (15) where from we may find that

$$\mathcal{H}_2(\sigma) = 2K_{1,1}(\sigma) = 2 \cos \sigma \sum_m^{\neq} \frac{\cos m\sigma}{m} [(-1)^m - 1]$$

and from the definition $(1)_2$

$$\mathcal{H}_2(\sigma) = \frac{4}{\sin \sigma} \sum_m^{\neq} \frac{1 - (-1)^m}{m^2 - 4} \sin m\sigma$$

Evaluation of an integral by the form

$$I = \frac{1}{\pi} \int_a^b \frac{f(t)}{t - x} dt \quad (22)$$

where $f(t)$ is a given hölderian function on the interval $[a, b]$ may be done using the Glauert's substitution $(t, x) \rightarrow (\theta, \sigma)$

$$\begin{aligned} t &= c + e \cos \theta, & x &= c + e \cos \sigma \\ c &= \frac{1}{2}(a + b), & e &= \frac{1}{2}(b - a) \end{aligned}$$

Let's consider the expansion of the function f into a cosines series

$$f(x) = f_0 + \sum_{n=1}^{+\infty} f_n \cos n\sigma$$

with the coefficients f_n defined by

$$f_0 = \frac{1}{\pi} \int_0^\pi F(\theta) d\theta, \quad F_n = \frac{2}{\pi} \int_0^\pi F(\theta) \cos n\theta d\theta.$$

Then, taking into account the Glauert formulas (10), (14) and (13) the above integral (22) becomes

$$I = f_0 \mathcal{H}_1(\sigma) + \sum_{n=1}^{+\infty} f_n K_{1,n}(\sigma)$$

where $\mathcal{H}_1(\sigma)$ and $K_{1,n}(\sigma)$ will be replaced by (20) and (16) respectively.

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