APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY DISCRETE LINEAR POSITIVE OPERATORS

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Abstract

We introduce a class of discrete linear positive operators in the space of real continuous functions of two variables defined on the square $[0,1]\times[0,1]$*. These operators are associated to an arbitrary set of nodes* (x_i, y_j), $i=0,\ldots,m$, $j=0,\ldots,n$, *and preserve the linear functions.*

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1. Introduction

The most results concerning the approximation of continuous functions on the compact domain $D=[0,1]\times[0,1]$ by polynomials use some linear positive operators which preserve the linear functions. Some discrete operators are of the form

$$T_{m,n}(f)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n} f(x_i,y_j)u_i(x)v_j(y), \tag{1}$$

where p_i are polynomials of degree m and q_j are polynomials of degree n.

An example is Bernstein operator

$$B_{m,n}(f)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n} f\Big(\frac{i}{m},\frac{j}{n}\Big)\binom{m}{i}x^i(1-x)^{m-i}\binom{n}{j}y^j(1-y)^{n-j}.$$

This operator has the well known properties:

$$\begin{aligned}
&B_{m,n}(e_0)(x,y)=1,\\
&B_{m,n}(e_1)(x,y)=x,\\
&B_{m,n}(h_1)(x,y)=y,\\
&B_{m,n}(e_2)(x,y)=x^2+\frac{1}{m}x(1-x),\\
&B_{m,n}(h_2)(x,y)=y^2+\frac{1}{n}y(1-y),
\end{aligned}$$

where $e_0(x,y)=1,\ e_1(x,y)=x,\ e_2(x,y)=x^2,\ h_1(x,y)=y$, and $h_2(x,y)=y^2$.

The aim of this paper is to give a characterization of those operators (1) for which there exist the real constants a and b such that

$$T_{m,n}(e_0)(x,y)=1, \tag{2}$$
$$T_{m,n}(e_1)(x,y)=x, \tag{3}$$
$$T_{m,n}(h_1)(x,y)=y, \tag{4}$$
$$T_{m,n}(e_2)(x,y)=x^2+ax(1-x), \tag{5}$$
$$T_{m,n}(h_2)(x,y)=y^2+by(1-y). \tag{6}$$

2. Main results

Let $\Delta_1=\{x_0=0<x_1<x_2<...<x_m=1\}$, $\Delta_2=\{y_0=0<y_1<y_2<...<y_n=1\}$ and $\Delta=\Delta_1\times\Delta_2$ be the fixed sets of nodes. For two sets of polynomials $P=\{p_1,p_2,...,p_{m-1}\}$ and $Q=\{q_1,q_2,...,q_{n-1}\}$ satisfying

$$\sum_{i=1}^{m-1}p_i(x)=1,\ \text{for all } x\in[0,1],\ \sum_{j=1}^{n-1}q_j(y)=1,\ \text{for all } y\in[0,1]$$

and for three real numbers $\alpha,\beta,A,\ A\neq 0$, we will build an operator which satisfies (2)-(6). First we construct the following polynomials:

$$u_0(x)=A(1-x)[1-(1-\alpha)x\sum_{k=1}^{m-1}\frac{1}{x_k}p_k(x)], \tag{7}$$
$$u_i(x)=A\frac{1-\alpha}{x_i(1-x_i)}x(1-x)p_i(x),\ i=1,2,\ldots,m-1,$$
$$u_m(x)=Ax\big[1-(1-\alpha)(1-x)\sum_{k=1}^{m-1}\frac{1}{1-x_k}p_k(x)\big],$$
$$v_0(y)=\frac{1}{A}(1-y)\big[1-(1-\beta)y\sum_{k=1}^{n-1}\frac{1}{y_k}q_k(y)\big],$$
$$v_j(y)=\frac{1}{A}\frac{1-\beta}{y_j(1-y_j)}y(1-y)q_j(y),\ j=1,2,\ldots,n-1,$$
$$v_n(y)=\frac{1}{A}y\big[1-(1-\beta)(1-y)\sum_{k=1}^{n-1}\frac{1}{1-y_k}q_k(y)\big].$$

Then we consider the operator $T_{P,Q,\alpha,\beta,A}$ defined on $C([0,1]\times[0,1])$, with values in the set of polynomials in two variables

$$T_{P,Q,\alpha,\beta,A}(f)(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}f(x_i,y_j)u_i(x)v_j(y) \tag{8}$$

Lemma 1. *The operator $T_{P,Q,\alpha,\beta,A}$ defined by* (8) *has the properties*

$$\begin{aligned}
T(e_0)(x,y) &= 1,\\
T(e_1)(x,y) &= x,\\
T(h_1)(x,y) &= y,\\
T(e_2)(x,y) &= x^2+\alpha x(1-x),\\
T(h_2)(x,y) &= y^2+\beta y(1-y).
\end{aligned}$$

Proof: We have

$$\begin{aligned}
u_i(x) &= A(1-\alpha)x(1-x)p_i(x)\left(\frac{1}{x_i}+\frac{1}{1-x_i}\right),\\
x_iu_i(x) &= A\frac{(1-\alpha)x(1-x)}{1-x_i}p_i(x),\\
x_i^2u_i(x) &= A(1-\alpha)x(1-x)\left(\frac{1}{1-x_i}-1\right)p_i(x),\ i=1,2,\ldots,m-1
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{m-1}u_i(x) &= A-u_0(x)-u_m(x),\\
\sum_{i=1}^{m-1}x_iu_i(x) &= Ax-u_m(x),\\
\sum_{i=1}^{m-1}x_i^2u_i(x) &= Ax-u_m(x)-A(1-\alpha)x(1-x)=Ax^2+A\alpha x(1-x)-u_m(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{i=0}^{m}u_i(x) &= A\\
\sum_{i=1}^{m}x_iu_i(x) &= Ax\\
\sum_{i=1}^{m-1}x_i^2u_i(x) &= A[x^2+\alpha x(1-x)]
\end{aligned}$$

Similarly we obtain that

$$\sum_{j=0}^{n} v_j(y) = \frac{1}{A},$$
$$\sum_{j=1}^{n} y_j v_j(y) = \frac{1}{A} y,$$
$$\sum_{j=1}^{n} y_j^2 v_j(y) = \frac{1}{A}[y^2 + \beta y(1-y)].$$

A simple calculation show that the conclusion of lemma is true.

In what follows we will consider only positive operators. Let $\mathcal{P}_{m,n}$, $m, n \geq 2$, denote the set of polynomials in two variables, of degree m in x and degree n in y and let $\mathcal{P}_s^k$ denote the linear space

$$\mathcal{P}_s^k = \{(p_1, p_2, ..., p_k), \text{ with } p_1, p_2, ..., p_k \text{ polynomials of degree } s\}.$$

An element $(p_1, p_2, ..., p_k) \in \mathcal{P}_s^k$ is said to be *admissible if*

$$p_i(x) \geq 0, \ i = 1, 2, \ldots, k \ \text{ and } \sum_{i=1}^{k} p_i(x) = 1 \text{ for all } x \in [0, 1].$$

For an admissible element $\mathbf{p} = (p_1, p_2, ..., p_{m-1}) \in \mathcal{P}_{m-2}^{m-1}$ we will define $\alpha(\mathbf{p})$ as

$$\alpha(\mathbf{p}) = \max \left\{ 1 - \frac{1}{m_1}, 1 - \frac{1}{m_2} \right\},$$

where

$$m_1 = \max \left\{ x \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(x), \ x \in [0, 1] \right\},$$
$$m_2 = \max \left\{ (1-x) \sum_{i=1}^{m-1} \frac{1}{1-x_i} p_i(x), \ x \in [0, 1] \right\}.$$

We remark that

$$m_1 \geq \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(1) > \sum_{i=1}^{m-1} p_i(1) = 1.$$

and $m_2 > 1$.

Similarly, for an admissible element $\mathbf{q} = (q_1, q_2, ..., q_{n-1}) \in \mathcal{P}_{n-2}^{n-1}$, we will define $\beta(\mathbf{q})$ as

$$\beta(\mathbf{q}) = \max \left\{ 1 - \frac{1}{n_1}, 1 - \frac{1}{n_2} \right\},$$

where

$$n_1 = \max\Big\{y\sum_{j=1}^{n-1}\frac{1}{y_j}q_j(y),\ y\in[0,1]\Big\},$$

$$n_2 = \max\Big\{(1-y)\sum_{j=1}^{n-1}\frac{1}{1-y_j}q_j(y),\ y\in[0,1]\Big\}.$$

Theorem 1. *Let us consider the admissible elements* $\mathbf{p}$ *and* $\mathbf{q}$*, where*

$$\mathbf{p} = (p_1, p_2, \dots, p_{m-1}) \in \mathcal{P}_{m-2}^{m-1}$$

and

$$\mathbf{q} = (q_1, q_2, \dots, q_{n-1}) \in \mathcal{P}_{n-2}^{n-1}.$$

Then the polynomials

$$u_0(x) = A(1-x)[1-(1-\alpha(\mathbf{p}))x\sum_{i=1}^{m-1}\frac{1}{x_i}p_i(x)],$$

$$u_i(x) = A\frac{1-\alpha(\mathbf{p})}{x_i(1-x_i)}x(1-x)p_i(x),\ i=1,2,\dots,m-1,$$

$$u_m(x) = Ax[1-(1-\alpha(\mathbf{p}))(1-x)\sum_{i=1}^{m-1}\frac{1}{1-x_i}p_i(x)],$$

$$v_0(y) = \frac{1}{A}(1-y)[1-(1-\beta(\mathbf{q}))y\sum_{j=1}^{n-1}\frac{1}{y_j}q_j(y)],$$

$$v_j(y) = \frac{1}{A}\frac{1-\beta(\mathbf{q})}{y_j(1-y_j)}y(1-y)q_j(y),\ j=1,2,\dots,n-1,$$

$$v_n(y) = \frac{1}{A}y[1-(1-\beta(\mathbf{q}))(1-y)\sum_{j=1}^{n-1}\frac{1}{1-y_j}q_j(y),$$

where A *is an arbitrary positive constant, are nonnegative on* $[0,1]$*. Moreover, the operator* $T : C([0,1]\times[0,1] \to \mathcal{P}_{m,n}$*,*

$$T(f)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} f(x_i,y_j)u_i(x)v_j(y),$$

is positive and has the properties

$$T(e_0)(x,y) = 1,$$
$$T(e_1)(x,y) = x, \quad T(h_1)(x,y) = y,$$
$$T(e_2)(x,y) = x^2 + \alpha(\mathbf{p})x(1-x), \quad T(h_2)(x,y) = y^2 + \beta(\mathbf{q})y(1-y).$$

Proof: From Lemma 1 we conclude that it is enough to prove that T is positive. Obviously

$$u_i(x) \geq 0, \text{ for } x \in [0,1], \; i = 1, 2, \ldots, m-1, \text{ and}$$
$$v_j(y) \geq 0, \text{ for } y \in [0,1], \; j = 1, 2, ..., n-1.$$

The inequality $u_0(x) \geq 0$ for $x \in]0,1]$ is equivalent to

$$\alpha(\mathbf{p}) \geq 1 - \frac{1}{x \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(x)},$$

which is true from the definition of $\alpha(\mathbf{p})$. Inequality $u_m(x) \geq 0$ for $x \in [0,1[$ is equivalent to

$$\alpha(\mathbf{p}) \geq 1 - \frac{1}{(1-x) \sum_{i=1}^{m-1} \frac{1}{1-x_i} p_i(x)},$$

which is also true from the definition of $\alpha(\mathbf{p})$. Similarly for v_0 and v_n.

In the case of theorem we will said that the operator T is *generated* by $\mathbf{p}$ and $\mathbf{q}$.

In the next theorem we show that each linear positive operator $T : C([0,1] \times [0,1]) \to \mathcal{P}_{m,n}$, having the form (1) and verifying properties (2)-(6), is generated by an admissible element $\mathbf{p} \in \mathcal{P}_{m-2}^{m-1}$ and an admissible element $\mathbf{q} \in \mathcal{P}_{n-2}^{n-1}$.

Theorem 2. *Let* $T : C([0,1] \times [0,1] \to \mathcal{P}_{m,n}$,

$$T(f)(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} f(x_i, y_j) u_i(x) v_j(y)$$

be a linear positive operator having the properties

$$T(e_0)(x,y) = 1,$$
$$T(e_1)(x,y) = x, \; T(h_1)(x,y) = y,$$
$$T(e_2)(x,y) = x^2 + ax(1-x), \; T(h_2)(x,y) = y^2 + by(1-y).$$

Then there exist the admissible elements $\mathbf{p} = (p_1, p_2, \ldots, p_{m-1}) \in \mathcal{P}_{m-2}^{m-1}$, $\mathbf{q} = (q_1, q_2, \ldots, q_{n-1}) \in \mathcal{P}_{n-2}^{n-1}$ *and a constant* $A > 0$, *such that* $\alpha(\mathbf{p}) \leq a$,

$\beta(\mathbf{q}) \leq b$ and

$$\begin{aligned}
u_0(x) &= A(1-x)\big[1-(1-a)x\sum_{i=1}^{m-1}\frac{1}{x_i}p_i(x)\big],\\
u_i(x) &= A\frac{1-a}{x_i(1-x_i)}x(1-x)p_i(x),\ i=1,2,\dots,m-1,\\
u_m(x) &= Ax\big[1-(1-a)(1-x)\sum_{i=1}^{m-1}\frac{1}{1-x_i}p_i(x)\big],\\
v_0(y) &= \frac{1}{A}(1-y)\big[1-(1-b)y\sum_{j=1}^{n-1}\frac{1}{y_j}q_j(y)\big],\\
v_j(y) &= \frac{1}{A}\frac{1-b}{y_j(1-y_j)}y(1-y)q_j(y),\ j=1,2,\dots,n-1,\\
v_n(y) &= \frac{1}{A}y\big[1-(1-b)(1-y)\sum_{j=1}^{n-1}\frac{1}{1-y_j}q_j(y)\big].
\end{aligned}$$

Proof: From $T(e_0)(x,y)=1$ we deduce that

$$\sum_{i=0}^{m}u_i(x)\sum_{j=0}^{n}v_j(y)=1,$$

whence $\sum_{i=0}^{m}u_i(x)$ and $\sum_{j=0}^{n}v_j(y)$ are constant. Let

$$A=\sum_{i=0}^{m}u_i(x). \tag{9}$$

Then,

$$\sum_{j=0}^{n}v_j(y)=\frac{1}{A}. \tag{10}$$

From $T(e_1)(x,y)=x$ we obtain

$$\sum_{i=1}^{m}x_iu_i(x)=Ax, \tag{11}$$

and from the positivity of the polynomials u_i, by taking $x=0$, we deduce that

$$u_i(x)=xu_i^{[1]}(x),\ i=1,2,\dots,n.$$

and equality (11) becomes

$$\sum_{i=1}^{m}x_iu_i^{[1]}(x)=A. \tag{12}$$

Further, from $T(e_2)(x,y) = x^2 + ax(1-x)$ we obtain

$$\sum_{i=1}^{m} x_i^2 u_i^{[1]}(x) = A[x + a(1-x)]. \tag{13}$$

and by subtracting (13) from (12) one gets

$$\sum_{i=1}^{m-1} x_i(1-x_i)u_i^{[1]}(x) = A(1-a)(1-x). \tag{14}$$

We remark that equality (14) implies $a < 1$.

For $x = 1$ we have

$$u_i^{[1]}(x) = (1-x)u_i^{[2]}(x),\ i = 1, 2, \ldots, m-1.$$

Then

$$u_i(x) = x(1-x)u_i^{[2]}(x),\ i = 1, 2, ..., m-1 \tag{15}$$

and (14) becomes

$$\sum_{i=1}^{m-1} x_i(1-x_i)u_i^{[2]}(x) = A(1-a). \tag{16}$$

For $i = 1, 2, \ldots, m-1$ we define

$$p_i(x) = \frac{x_i(1-x_i)}{A(1-a)} u_i^{[2]}(x) = \frac{x_i(1-x_i)}{A(1-a)} \frac{u_i(x)}{x(1-x)}. \tag{17}$$

Then p_i are positive polynomials of degree $m-1$, and from (16) we obtain

$$\sum_{i=1}^{m-1} p_i(x) = 1.$$

From (17) we obtain

$$u_i(x) = A(1-a)x(1-x)\frac{p_i(x)}{x_i(1-x_i)},\ i = 1, 2, \ldots, m-1.$$

Further, from (11) we deduce that

$$u_m(x) = A\Big[1 - (1-a)(1-x)\sum_{i=1}^{m-1} \frac{p_i(x)}{1-x_i}\Big]$$

and from (9) we obtain

$$u_0(x) = A\Big[1 - (1-a)x\sum_{i=1}^{m-1} \frac{p_i(x)}{x_i}\Big].$$

But $u_0(x) \geq 0$, therefore

$$x \sum_{i=1}^{m-1} \frac{p_i(x)}{x_i} \leq \frac{1}{1-a} \text{ and}$$

$$m_1 = \max \left\{ x \sum_{i=1}^{m-1} \frac{1}{x_i} p_i(x),\ x \in [0,1] \right\} \leq \frac{1}{1-a}.$$

Analogous, from $u_m(x) \geq 0$ we obtain

$$m_2 = \max \left\{ (1-x) \sum_{i=1}^{m-1} \frac{1}{1-x_i} p_i(x),\ x \in [0,1] \right\} \leq \frac{1}{1-a},$$

therefore $\alpha(\mathbf{p}) \leq a$.

Similarly, we can conclude that, in the case when

$$q_j(y) = \frac{Ay_j(1-y_j)}{(1-b)} \frac{v_j(y)}{y(1-y)},\ j = 1, 2, \ldots, n-1,$$

the following equalities hold:

$$\sum_{j=1}^{n-1} q_j(y) = 1,$$

$$v_j(y) = \frac{(1-b)y(1-y)q_j(y)}{Ay_j(1-y_j)},\ j = 1, 2, \ldots, n-1,$$

$$v_n(y) = \frac{1}{A}[1 - (1-b)(1-y) \sum_{j=1}^{n-1} \frac{q_j(y)}{1-y_j}],$$

$$v_0(y) = \frac{1}{A}[1 - (1-b)y \sum_{j=1}^{n-1} \frac{q_j(y)}{y_j}],$$

$$\beta(\mathbf{q}) \leq b.$$

Remark 1. *In the hypotheses of the theorem, if $\alpha(\mathbf{p}) < a$ or $\beta(\mathbf{p}) < b$, then the operator T^* generated by $\mathbf{p}$ and $\mathbf{q}$ is better than T since*

$$T^*(e_0)(x,y) = 1,$$
$$T^*(e_1)(x,y) = x, \quad T^*(h_1)(x,y) = y,$$
$$T^*(e_2)(x,y) = x^2 + \alpha(\mathbf{p})x(1-x), \quad T^*(h_2)(x,y) = y^2 + \beta(\mathbf{q})y(1-y).$$

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