ON THE EXPONENTIAL DIOPHANTINE EQUATION

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Abstract

The aim of this paper is to give all nonnegative solutions (x, y, z) *to the equation* $2^x + p^y = 1009^z$, *where* p *is a positive rational prime number with* $3 \leq p \leq 997$ *(167 equations)*

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1. **Introduction**

It is known that the equation

$$a^x + b^y = c^z \tag{1}$$

where a, b, c are prime numbers, has only finitely many solutions, but there is no algorithm to compute all the solutions (x, y, z). Some particular cases were treated: Nagell [8] found all solutions for $\max(a, b, c) = 7$, Makowski [5], Hadano [4], Uchiyama [12], Qi Sun and Xiaoming Zhou [10] and Xiaozhuo Yang [13] determined all solutions for $11 \leq \max(a, b, c) \leq 23$. Cao [3] found all solutions for $29 \leq \max(a, b, c) \leq 97$ (60 solutions in total).

The aim of this paper is to give all nonnegative solutions to equation (1) for $a = 2, c = 1009$ (1009 representing the first prime number > 1000) and b is a rational prime number, $3 \leq b \leq 997$. The main results are given by the following

Theorem 1. *The only equations* $2^x + p^y = 1009^z$, *with p rational prime,* $3 \leq p \leq 1009$, *which admit nonnegative solutions* (x, y, z), *are:*

- $2^x + 881^y = 1009^z$, *which has the solution* $(x, y, z) = (7, 1, 1)$.
- $2^x + 977^y = 1009^z$, *which has the solution* $(x, y, z) = (5, 1, 1)$.

The proof for Theorem 1 is given in subsection 3.

2. **Preliminaries**

Below we present some theorems which establish the maximum number of solutions for the equation (1). Theorem 2 shows that the Catalan's equation has only one solution and gives this solution.

Theorem 2. ([1, 6, 7]) *Equation (named Catalan's equation)*

$$a^x - b^y = 1 \tag{2}$$

has no solutions in integers $a, b, x, y > 1$ *other than* $3^2 - 2^3 = 1$.

Theorem 3 is concerned with equation $a^x + b^y = p^z$, p rational prime, and proves that this equation has at most two solutions, except some particular equations, which have three solutions.

Theorem 3. ([9, Lemma 6, p.163]) *The equation*

$$a^x + b^y = p^z \tag{3}$$

has at most one solution when the parity of x *and* y *are preassigned, except for three choices of* (a, b, p) *taking* $(a < b) : (3, 5, 2), (3, 13, 2), (3, 10, 13)$.

3. **Proofs of the main results**

We give the proof for Theorem 1, which treats the equation

$$2^x + p^y = 1009^z \tag{4}$$

If $y = 0$, the equation (4) has no solutions, due to Theorem 2. So we consider $y > 0$. Many of the equations (4) have no solutions (x, y, z) due to:

Lemma 1. *If* $p \equiv 1, 5, 7, 11, 13, 19, 23, 25, 29, 31, 35, 37, 43$ *(mod 48) or* $p \equiv 1$ *(mod 7), then the equation (4) has no nonnegative solutions* (x, y, z).

Proof: It is obvious, taking the equation (4) mod 48 and mod 7, respectively.

Thus, we discuss the following equations, in which p doesn't comply with the conditions in Lemma 1:

i/ $2^x + p^y = 1009^z$, where $p \in \{89, 137, 233, 521, 569, 761, 809, 857\}$; by taking the equation mod 48, it results that $x = 3$; by taking the equation mod 7, it results that there are no solutions.

ii/ $2^x + p^y = 1009^z$, where $p \in \{191, 431, 863\}$; by taking the equation mod 48, it results that $x = 1$; by taking the equation mod 7, it results that there are no solutions.

iii/ $2^x + 3^y = 1009^z$, which has no solutions by taking the equation mod 60.

iv/ $2^x + 17^y = 1009^z$, which has no solutions by taking the equation mod 3, which leads to x and y odd, and mod 17, which becomes impossible with x and y odd.

v/ $2^x + 41^y = 1009^z$; by taking the equation mod 48, it results that x=3, so the equation becomes $8 + 41^y = 1009^z$, which has no solutions by taking the equation mod 7.

vi/ $2^x + 47^y = 1009^z$; by taking the equation mod 48, it results that x=1, so the equation becomes $2 + 47^y = 1009^z$, which has no solutions, by taking the equation mod 3 and mod 13.

vii/ $2^x + 257^y = 1009^z$; by taking the equation mod 3, it results that x and y are odd; by taking the equation mod 32, one gets $x \geq 5$ and z is even; by taking the equation mod 5, it results that $x \equiv 3 (\text{mod } 4)$ and $y \equiv 3 (\text{mod } 4)$; by taking the equation mod 13, it results that $x \equiv 3 (\text{mod } 12)$, $y \equiv 11 (\text{mod } 12)$ and $z \equiv 2 (\text{mod } 4)$; by taking the equation mod 7, one concludes that there are no solutions in this case.

viii/ $2^x + 353^y = 1009^z$; by taking the equation mod 3, it results that x and y are odd; by taking the equation mod 7, it results that x≡1 (mod 3) and y≡3 (mod 6); by taking the equation mod 9 and taking into account the relations above, it results that x≡1 (mod 6); by taking mod 13 and taking into account the relations obtained till now, the equation has no solutions.

ix/ $2^x + 383^y = 1009^z$; by taking the equation mod 24, it results that x = 1 and y is odd, so the equation becomes $2 + 383^y = 1009^z$; by taking this equation mod 5, it results that $y \equiv 3 (\text{mod } 4)$ and z is odd; by taking the equation mod 13, one concludes that there are no solutions in this case.

x/ $2^x + 401^y = 1009^z$; by taking the equation mod 7, it results that x≡2 (mod 3) and y≡2 (mod 3); by taking the equation mod 9 and taking into account the relations above, it results that the equation has no solutions.

xi/ $2^x + 479^y = 1009^z$; by taking the equation mod 48, it results that x=1; so the equation becomes $2 + 479^y = 1009^z$; by taking the equation mod 9, it results that y≡3 (mod 6); taking this relation into consideration and taking the equation mod 13, it results that the equation has no solutions.

xii/ $2^x + 593^y = 1009^z$; by taking the equation mod 3, it results that x and y are odd; by taking the equation mod 9, it results that x≡1 (mod 6); taking the equation mod 13, it results that the equation has no solutions.

xiii/ $2^x + 641^y = 1009^z$; by taking the equation mod 32, it results that one has either $x = 4$ and z odd or $x \geq 5$ and z even; in both cases, by taking the equation mod 5, it results that the equation has no solutions.

xiv/ $2^x + 719^y = 1009^z$; by taking the equation mod 48, it results that x=1, so the equation becomes $2 + 719^y = 1009^z$; by taking this equation mod 1009, it results that the equation has no solutions.

xv/ $2^x + 881^y = 1009^z$; by taking the equation mod 3, it results that x and y are odd; the equation has the solution (x,y,z)=(7,1,1) and it is the only one, due to Theorem 3.

xvi/ $2^x + 929^y = 1009^z$; by taking the equation mod 3, it results that x and y are odd; by taking the equation mod 7, it results that x$\equiv$ 1(mod 6) and y$\equiv$ 3(mod 6); taking now the equation mod 13, it results that it has no solutions.

xvii/ $2^x + 977^y = 1009^z$; by taking the equation mod 3, it results that x and y are odd; the equation has the solution (x,y,z)=(5,1,1) and it is the only solution, due to Theorem 3.

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