APPROXIMATE SADDLE POINTS PORTFOLIOS

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Abstract

We introduce the concept of saddle points portfolio and we establish a connection between this concept and the approximate minima.

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1. **Introduction**

The reinsurance problem appears at first sight to be a problem which can be analyzed in terms of classical economic theory, if the objectives of the companies have been formulated in an operational manner by the help of Bernoulli's utility concept: it must not be maximized the expected gain, but the expected utility of the gain [3]. However, closer investigations show that the economic theory is only relevant part of the way. Then the problem becomes a problem of cooperation between parties who have conflicting interests, and who are free to form and break any coalitions which may serve their particular interests [12]. Classical economic theory is powerless when it comes to analyze such problems. In the last decades it is shown that there are many possibilities to study and to explain the apparently chaotic situation by helping the games theory or convex analysis.

In this paper we present some new concepts who refine some result about saddle points portfolio and establish a connection between this concept and the almost ϵ- weak minimum portfolio. We use and extend the results established by Dutta and Vetriviel [2001] . All these results help us to establish some new concept for the portfolios in a reinsurance market for nonconvex optimization problem. A basic result of convex analysis is the Fundamental Theorem on Convex Functions [4] , [13] which states that an efficient solutions of a convex vector optimization problem necessarily minimizes a linear combination of objectives functions. In this paper we also use and extend results like efficient portfolios or $\epsilon-$efficient portfolios based on Ghica [2008, 2009] .

If we see N as a group of n reinsurers, having preferences $\geq_i, i \in N$, over a suitable set of random variables denoted by R, or gambles with realizations (outcomes) in some $A \subseteq R$, we represent these preferences by von Neumann-Morgenstern expected utility, meaning that there is a set of continuous utility functions $u_i : R \to \mathbb{R}$, such that $X \geq_i Y$ if and only if $Eu_i(X) \geq Eu_i(Y)$, where by the symbol E we denoted the mean operator. We assume monotonic preferences, and risk aversion, so that, we have $u_i^{'}(w) > 0$, $u_i^{''}(w) \leq 0$ for all w in the relevant domains [5]. In some of the cases we shall also require strict risk aversion, meaning strict concavity for some u_i. For a better understanding we presume that each agent is endowed with a random variable payoff X_i called initial portfolio. More precisely, there exists a probability space $(\Omega,\ \mathcal{K},\ P)$ such that we have the payoff $X_i(\omega)$ when $\omega \in \Omega$ occurs and, more, we the both expected values and variances exist for all these initial portfolios, which means that all $X_i \in L^2(\Omega,\ \mathcal{K},\ P)$ [6]. Because every agent can negotiate any affordable contracts then we will have a new set of random variables Y_i, $i \in N$, representing the final portfolios. We say if the following condition exists $\sum_{i=1}^{n} Z_i \leq \sum_{i=1}^{n} X_i = X_N$ then an allocation $Z = (Z_1, Z_2, ..., Z_n)$ is call feasible [1].

The paper is organized as follows: In Section 2 we define and characterize concepts like the Lagrangian function, an approximative saddle points portfolio or $\epsilon-$approximative saddle points portfolio and in Section 3 we give some condition for an approximative saddle points portfolios in a reinsurance market for the nonconvex optimization problem with respect to scalarization.

2. **Preliminaries**

Consider the following optimization problem:

$$\begin{array}{c}\min\left(Eu_1(Z_1),\ Eu_2(Z_2),\ ...,\ Eu_n(Z_n)\right)\\ Z \in \mathcal{Z}\end{array} \tag{1}$$

In this paper the feasible set of portfolios $\mathcal{Z}$ in (1) will be given in the following form: $\mathcal{Z} = \left\{ Z | \sum\limits_{i \in N} Z_i \leq \sum\limits_{i \in N} X_i \right\}$ where $(X_1, ..., X_n)$ is the initial portfolio X.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in \mathbb{R}^n$ be a tolerance vector satisfying $\varepsilon_i \geq 0$, $i = \overline{1, n}$

Definition 1. *The Lagrangian function associated with the problem (1) is defined as*

$$L(Z, \tau, \mu) = \sum_{i \in N} \tau_i Eu_i(Z_i) + \sum_{i \in N} \mu_i (Z_i - X_i),\ \ \tau, \mu \in \mathbb{R}^n$$

Definition 2. $Y \in \mathcal{Z}$ *is an efficient portfolio (Pareto portfolio) if for* $Z \in \mathcal{Z}$, *from* $Eu_i(Z_i) \leq Eu_i(Y_i)$, $i \in N$ *it follows that* $Eu_i(Z_i) = Eu_i(Y_i)$, for all $i \in N$.

Definition 3. $Y \in \mathcal{Z}$ *is a weakly efficient portfolio (Slater portfolio) if there is no* $Z \in \mathcal{Z}$ *such that* $Eu_i(Z_i) < Eu_i(Y_i)$, $i \in N$.

Definition 4. *Given any $\varepsilon \geq 0$, a triplet $\left(Z^0, \tau^0, \mu^0\right) \in \mathcal{Z} \times \mathbb{R}_+^n \times \mathbb{R}_+^n$ is said to be an approximative saddle point portfolio or $\varepsilon-$ saddle point portfolio of the Lagrangian function L if*

$$L\left(Z^0, \tau^0, \mu\right) - \varepsilon \leq L\left(Z^0, \tau^0, \mu^0\right) \leq L\left(Z, \tau^0, \mu^0\right) + \varepsilon$$

for $\forall Z \in \mathcal{Z}$ and $\mu \in \mathbb{R}_+^n$.

For $\varepsilon \geq 0$ we define the set of portfolios $\mathcal{Z}_\varepsilon = \left\{ Z | \sum\limits_{i\in N} Z_i - \sum\limits_{i\in N} X_i \leq \varepsilon \right\}$.

Definition 5. *A portfolio $Y \in \mathcal{Z}_\varepsilon$ is said to be an almost ε - weak minimum for (1) if there exists no $Z \in \mathcal{Z}$ such that*

$$Eu_i(Z_i) < Eu_i(Y_i) - \varepsilon, \ \forall i = \overline{1, n}.$$

From this definition we can note the set $\mathcal{Z}_\varepsilon$ is the set of the approximative feasible portfolios.

3. Approximate saddle points portfolios

Theorem 1. *Consider the minimization problem (1) and let $Z^0 \in \mathcal{Z}$ an ε -weak minimum for (1). Let us assume that the Slater portfolio qualification hold in (1). Then there exists $\left(\tau^0, \mu^0\right)$ such that $\left(Z^0, \tau_0, \mu_0\right)$ is an $\varepsilon-$saddle point portfolio of the Lagrangian function L.*

Proof: We have that $Z_0 \in \mathcal{Z}$ an ε -weak minimum for (1), the following system

$$Eu_i(Z_i) < Eu_i(Z_i^0) - \varepsilon, \text{ with} \sum_{i\in N} Z_i - \sum_{i\in N} X_i \leq \varepsilon, \forall i = \overline{1, n}$$

has no solutions. Hence by standard separation arguments we deduce that we can assume that it exists $\left(\tau^0, \mu^0\right)$ such that

$$\sum_{i\in N} \tau_i^0 \left(Eu_i(Z_i) - Eu_i(Z_i^0)\right) + \sum_{i\in N} \varepsilon \tau_i^0 + \sum_{i\in N} \mu_i^0 \left(Z_i - X_i\right) < 0, \ \forall Z \in \mathcal{Z} \quad (2)$$

If we assume the $\tau^0 = 0$ and applying the Slater constraints we prove easily that $\sum\limits_{i\in N} \mu_i^0 \left(Z_i - X_i\right) < 0$ contradiction with the relation (2). Hence $\tau^0 \neq 0$ and without loss of generality we can consider that $\sum\limits_{i\in N} \tau_i^0 = 1$. From this deduction we get

$$\sum_{i\in N} \tau_i^0 Eu_i(Z_i) + \varepsilon + \sum_{i\in N} \mu_i^0 \left(Z_i - X_i\right) < \sum_{i\in N} \tau_i^0 Eu_i(Z_i^0), \ \forall Z \in \mathcal{Z} \quad (3)$$

As $Z^0 \in \mathcal{Z}$ and $\mu^0 \in \mathbb{R}_+^n$ we have $\sum\limits_{i\in N} \mu_i^0 \left(Z_i^0 - X_i\right) \leq 0$, so, (2) can be formulated in this way:

$$\sum_{i\in N} \tau_i^0 Eu_i(Z_i) + \sum_{i\in N} \mu_i^0 \left(Z_i - X_i\right) + \varepsilon \geq \sum_{i\in N} \tau_i^0 Eu_i(Z_i^0) + \sum_{i\in N} \mu_i^0 \left(Z_i^0 - X_i\right), \forall Z \in \mathcal{Z}$$

And by the Lagrangian function we obtain:

$$L\left(Z^0,\tau^0,\mu^0\right) \leq L\left(Z,\tau^0,\mu^0\right)+\varepsilon, \forall Z \in \mathcal{Z}$$

For the second inequality we have that for $\forall \mu \in \mathbb{R}^n_+$ it is true that

$$\sum_{i\in N}\mu_i\left(Z_i^0 - X_i\right) \leq 0.$$

From this we can note that:

$$\sum_{i\in N}\tau_i^0 Eu_i(Z_i^0) + \sum_{i\in N}\mu_i\left(Z_i^0 - X_i\right) \leq \sum_{i\in N}\tau_i^0 Eu_i(Z_i^0)$$

$$\sum_{i\in N}\tau_i^0 Eu_i(Z_i^0) + \sum_{i\in N}\mu_i\left(Z_i^0 - X_i\right) \leq \sum_{i\in N}\tau_i^0 Eu_i(Z_i^0) + \varepsilon - \varepsilon$$

$$\sum_{i\in N}\tau_i^0 Eu_i(Z_i^0) + \sum_{i\in N}\mu_i\left(Z_i^0 - X_i\right) - \varepsilon \leq \sum_{i\in N}\tau_i^0 Eu_i(Z_i^0) - \varepsilon$$

And applying the relation (3) by setting $Z = Z^0$ we have that $\sum\limits_{i\in N}\mu_i^0\left(Z_i^0 - X_i\right) + \varepsilon \geq 0$, so, we deduce that

$$\sum_{i\in N}\tau_i^0 Eu_i(Z_i^0) + \sum_{i\in N}\mu_i\left(Z_i^0 - X_i\right) - \varepsilon \leq \sum_{i\in N}\tau_i^0 Eu_i(Z_i^0) + \sum_{i\in N}\mu_i^0\left(Z_i^0 - X_i\right)$$

Hence, by the Lagrangian function and for all $\mu \in \mathbb{R}^n_+$ we have that

$$L\left(Z^0,\tau^0,\mu\right) - \varepsilon \leq L\left(Z^0,\tau^0,\mu^0\right)$$

The converse of the above theorem is not appear to be true but we have the following result by helping of the concept of solution of an almost ε-weak minimum.

Theorem 2. *Consider the optimization problem (1). Let $\varepsilon_1, \varepsilon_2 \geq 0$ be given. Let be a point portfolio $\left(Z^0,\tau_0,\mu_0\right) \in \mathcal{Z}\times\mathbb{R}^n_+\times\mathbb{R}^n_+$ such that Z^0 is an ε_1-minimum of $L\left(\cdot,\tau^0,\mu^0\right)$ over $\mathcal{Z}$ and let μ_0 is an ε_2-maximum of $L\left(Z^0,\tau^0,\cdot\right)$ over $\mathbb{R}^n_+$. Then Z^0 is an almost ε-weak minimum of (1) with $\varepsilon = \varepsilon_1 + \varepsilon_2$.*

Proof: We have the μ_0 is an ε_2-maximum of $L\left(Z^0,\tau^0,\cdot\right)$ over $\mathbb{R}^n_+$, hence we have for all $\mu \in \mathbb{R}^n_+$

$$L\left(Z^0,\tau^0,\mu\right) - \varepsilon_2 \leq L\left(Z^0,\tau^0,\mu^0\right)$$

or,

$$\sum_{i\in N}\left(\mu_i - \mu_i^0\right)\left(Z_i^0 - X_i\right) \leq \varepsilon_2, \forall \mu \in \mathbb{R}^n_+ \tag{4}$$

In the following we prove the $Z^0 \in \mathcal{Z}_{\varepsilon_2}$. On the contrary we assume that $Z^0 \notin \mathcal{Z}_{\varepsilon_2}$. So, $\sum\limits_{i\in N} Z_i - \sum\limits_{i\in N} X_i > \varepsilon_2$ and there exists a vector nonzero $p \in \mathbb{R}^n_+$ such that

$$\sum_{i\in N}p_i\varepsilon_2 - \sum_{i\in N}p_i\left(Z_i^0 - X_i\right) < 0$$

and we can consider $\sum\limits_{i\in N} p_i = 1$ without loss generality, the last relation becomes:

$$\varepsilon_2 - \sum_{i\in N} p_i \left(Z_i^0 - X_i\right) < 0$$

hence we have that

$$\sum_{i\in N} p_i \left(Z_i^0 - X_i\right) > \varepsilon_2$$

relation what is in contradiction with (4) and we can conclude that $Z^0 \in \mathcal{Z}_{\varepsilon_2}$. Hence we have that $Z^0 \in \mathcal{Z}_\varepsilon$ since $\varepsilon = \varepsilon_1 + \varepsilon_2$ and $\varepsilon_1 \geq 0$.

Since Z^0 is an ε_1-minimum of $L\left(\cdot, \tau^0, \mu^0\right)$ over $\mathcal{Z}$ we have

$$L\left(Z^0, \tau^0, \mu^0\right) \leq L\left(Z, \tau^0, \mu^0\right) + \varepsilon_1, \forall Z \in \mathcal{Z}$$

Hence we have the following relation

$$\sum_{i\in N} \tau_i^0 \left(Eu_i(Z_i) - Eu_i(Z_i^0)\right) + \sum_{i\in N} \mu_i^0 \left(Z_i - X_i\right) - \sum_{i\in N} \mu_i^0 \left(Z_i^0 - X_i\right) + \varepsilon_1 \geq 0,\ \forall Z \in \mathcal{Z}$$

We use the inequality $\sum\limits_{i\in N} \mu_i^0 \left(Z_i - X_i\right) \leq 0$ and the above relation becomes

$$\sum_{i\in N} \tau_i^0 \left(Eu_i(Z_i) - Eu_i(Z_i^0)\right) - \sum_{i\in N} \mu_i^0 \left(Z_i^0 - X_i\right) + \varepsilon_1 \geq 0,\ \forall Z \in \mathcal{Z} \tag{5}$$

From (4) for $\mu = 0$ we have $\sum\limits_{i\in N} \mu_i^0 \left(Z_i^0 - X_i\right) \geq -\varepsilon_2$ hence from (5) we obtain the following inequalities

$$\begin{aligned}
&\sum_{i\in N} \tau_i^0 \left(Eu_i(Z_i) - Eu_i(Z_i^0)\right) \geq \sum_{i\in N} \mu_i \left(Z_i^0 - X_i\right) - \varepsilon_1 \\
&\sum_{i\in N} \tau_i^0 \left(Eu_i(Z_i) - Eu_i(Z_i^0)\right) \geq -\varepsilon_2 - \varepsilon_1
\end{aligned}$$

Since $\varepsilon = \varepsilon_1 + \varepsilon_2$ we finally obtain $\sum\limits_{i\in N} \tau_i^0 \left(Eu_i(Z_i) - Eu_i(Z_i^0)\right) \geq \varepsilon$. We have $Z^0 \in \mathcal{Z}_\varepsilon$ and we claim that Z^0 is a almost ε-weak minimum of (1). If this not holds then $\exists Y \in \mathcal{Z}$ such that

$$Eu_i(Y_i) - Eu_i(Z_i^0) < -\varepsilon,\ \forall i \in N$$

But $\tau^0 \in \mathbb{R}_+^n$ such that $\sum\limits_{i\in N} \tau_i = 1$ we have

$$\sum_{i\in N} \tau_i^0 \left(Eu_i(Z_i) - Eu_i(Z_i^0)\right) < -\varepsilon$$

false with (5).

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