A CHARACTERIZATION OF DIRECTED P-GRAPHS

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Abstract

In this paper we introduce directed P-graphs and P_k-graphs, as extensions of their previously studied undirected counterparts. The main result of this research computes the minimum number of edges of directed P-graphs and describes the structure of such minimal graphs. We establish a connection between this minimal structure and integer partitions, which allows us to exactly compute the number of distinct, non-isomorphic minimal directed P-graphs. We also characterize the density of the more general directed P_k-graphs relative to the entire set of graphs.

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1. Introduction

This paper presents directed graph generalizations of the notions of P-graphs and P_k graphs described in [2] and [3]. In [2], the authors study a problem of minimum in graph theory: undirected P-graphs are introduced as graphs that have a path of length two between any two vertices; the minimum number of edges in such graphs is computed and the structure of the minimal undirected P-graphs is inferred. Results on particular classes of P-graphs have been derived earlier in [1] and [5].

Later, in [3], the P-graphs are extended in two directions to obtain P_k-graphs and Q_k-graphs. Briefly, in a P_k-graph, each k-tuple of vertices has a common neighbor, whereas in the case of Q_k-graphs, there is a path of length k between each two vertices. Bounding inequalities are derived for the minimum number of edges of undirected P_k and Q_k graphs. The authors also present theorems that describe the density of the undirected P_k and Q_k graphs within the set of all undirected graphs.

The purpose of this paper is to build on the research mentioned above. In particular, we investigate directed extensions of P-graphs and P_k-graphs. We derive the minimum number of edges in directed P-graphs, then discover and count the structures of such minimal graphs. We conclude with a density result similar with the ones for undirected graphs.

In the next section we will present the main results of our research and then we will conclude with a summary as well as some interesting ideas to explore as future work.

2. **Results**

Definition 1. *We say that the pair (x, y) - where x and y are two distinct vertices of a directed graph G - has "Property P^-" if there exists a third vertex z, as well as a directed edge from x to z and a directed edge from y to z. We say that G is a "Directed P-graph" if for every x and y distinct vertices of G, the pair (x, y) has Property P^-.*

Definition 2. *Let $G(V, E)$ be a directed graph. We say that the k-tuple $(x_1, \ldots, x_k)$ of distinct vertices from $V(G)$ has "Property P_k^-" if there exists a vertex y from $V(G)$, different from $x_1, x_2, \ldots, x_k$, such that there exist directed edges starting from each of $x_1, x_2, \ldots, x_k$ and ending y. We say that G is a "Directed P_k-graph" if for every $x_1, x_2, \ldots, x_k$ distinct vertices of G, then k-tuple $(x_1, \ldots, x_k)$ has Property P_k^-.*

Observation 1. For each positive integers k and n such that $n \geq k + 1$, there exist directed P_k-graphs with n vertices and there exist directed graphs with n vertices which are not directed P_k-graphs. To see this, it is enough to consider the directed complete graph K_n^* with $n(n-1)$ edges in the first case and a directed graph with n vertices and no edges in the second case.

Definition 3. *Let us denote by $a^-(n)$ the minimum number of edges of a directed P-graph. Similarly with the case of undirected P-graphs [**?**], we will give a theorem that computes $a^-(n)$ and describes the structure of minimal directed P-graphs with n vertices. We say that a directed P-graph with n vertices is minimal if it has $a^-(n)$ edges.*

Theorem 1. *We have $a^-(n) = 2n$ and any minimal P-directed graph G with n vertices has the following structure: there exist three vertices which form a complete directed subgraph with three vertices and six edges, and the rest of the vertices have in-degree $d^- = 0$ and out-degree $d^+ = 2$ and exactly two edges start from any such vertex towards two of the vertices of the complete directed subgraph with three vertices.*

Proof: Let $G(V, E)$ be a minimal directed P-graph with n vertices. First, we will prove that for every vertex x of G we have $d^+(x) \geq 2$. Let y be a vertex different than x. Then, the pair (x, y) has Property P^- and therefore there exists a vertex z, such that there are edges starting at x and at y which end at z. Also, the pair (x, z) has Property P^- and therefore there exists a third vertex t, different from x and z, such that there are edges starting at x and z

which end at t. In conclusion, there are at least two edges starting at x (xt and xz) or, equivalently, $d^+(x) \geq 2$.

For any directed graph we have the following relation between the number of edges and the degrees of the vertices:

$$|E(G)| = \sum_{x \in V(G)} d^-(x) = \sum_{x \in V(G)} d^+(x).$$

From the observations above, we can derive the following inequality:

$$a^-(n) = |E(G)| = \sum_{x \in V(G)} d^+(x) \geq 2|V(G)| = 2n.$$

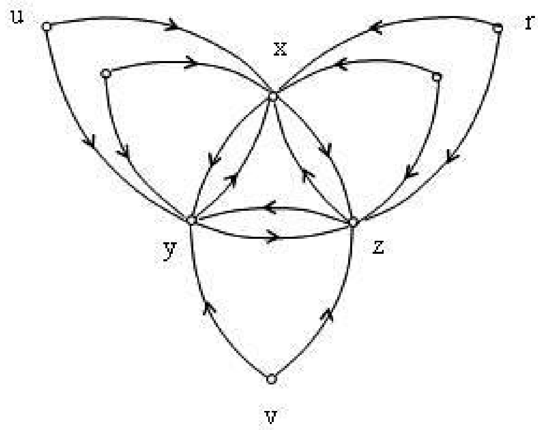

Figure 1. *The structure of a minimal P directed graph*

Now let us consider the directed graph G_1 with n vertices from Figure 1, which has the following structure: there exist three vertices that form a complete directed subgraph and the remaining vertices have $d^- = 0$ and $d^+ = 2$. It is easy to see that G_1 is a directed P-graph for which the three vertices with $d^- \neq 0$ form a complete directed subgraph and for which the remaining vertices start exactly two edges towards two of the vertices of the complete subgraph. The number of edges of G_1 is $2n$, and therefore we have: $a^-(n) \leq 2n$.

From the two inequalities derived above relative to $a^-(n)$, we obtain $a^-(n) = 2n$. Thus, the first part of the theorem is proved. Next, we are going to find the structure of the graphs that achieve this minimum.

It is easy to see that the only directed P-graph with three vertices is the directed complete graph. This graph has the structure described in the statement of the theorem and therefore for $n = 3$ the conclusion holds true. We are going to prove the general result by induction on n. In other words, let us assume that the minimal directed P-graphs with $n-1$ vertices have the structure in the statement of the theorem and let us prove the same for n.

Let G be a minimal directed P-graph with $n > 3$ vertices. From the result above, the number of edges of G is $2n$. We can see that G does not contain any vertices x with $d^-(x) = 1$. This holds true because, if we had $d^-(x) = 1$, we could suppress the edge that ends in x and we would obtain another directed P-graph, with less than $2n$ edges, which is a contradiction to the fact that $a^-(n) = 2n$. Therefore $d^-(x) = 0$ or $d^-(x) \geq 2$ for all vertices x of G.

If $d^-(x) \geq 2$ for all vertices x, then, from the fact that $\sum_x d^-(x)$ represents the number of edges of G, which is equal to $2n$, we infer that $d^-(x) = 2$ for all vertices x. We define a function f on the set of pairs of vertices of G with the values in the set of vertices of G in the following way: to each pair (x, y) we associate one and only one of the vertices z that make the pair (x, y) have Property P^-. Since G is a directed P-graph, f is well defined. Let us assume f can take the same value z for two distinct pairs of vertices. Therefore there exist at least three vertices from which we have edges ending in z, which is a contradiction to the fact that $d^-(z) = 2$. Therefore f is an injective function and thus, the number of vertices of G is less than or equal to the number of pairs of vertices of G, which is obviously false for $n > 3$. We conclude that it is not possible to have $d^-(x) \geq 2$ for all vertices x of G. Based on the above, there must exist a vertex u such that $d^-(u) = 0$.

In the beginning of the proof we showed that $d^+(x) \geq 2$ for all vertices x of a directed P-graph. Since G is a minimal directed P-graph, then G has $2n$ edges, and therefore we infer $d^+(x) = 2$ for all vertices x. In particular, there are two edges that start at u.

Let G_2 be the graph obtained from G by suppressing the vertex u and the two edges that start at u. Since there are no edges that end at u, it follows that all pairs of vertices of G_2 still have property P^- after u was removed and thus G_2 is also a directed P-graph. Moreover, G_2 has $2n - 2$ edges and $n - 1$ vertices, and thus G_2 is a minimal directed P-graph. According to the induction hypothesis, G_2 has the structure described in the statement of the theorem. Let x, y and z be the three vertices of G_2 that form the complete directed subgraph The pair (x, u) has Property P^- in G and therefore there exists a vertex of G such that there are edges starting at x and u which end at this vertex. Also according to the induction hypothesis, the only edges starting at x are ending at y and z. It follows that there is an edge starting at u and ending at either y or z. Because of the symmetry, we can assume there is an edge from u to y. Through a similar reasoning on the pair (y, u), we infer there is an edge starting at u that ends at either x or z. Because $d^+(u) = 2$, it follows that there are exactly two edges starting at u and which end at two of the vertices of the complete directed subgraph given by x, y, and z. If we combine this with the induction hypothesis, we get that G has the structure described in the statement of the theorem and therefore the induction step is completely proved. This concludes the proof of the theorem.

Next we will count the number of minimal directed P-graphs with n vertices labeled $\{1, \ldots, n\}$. We have the following result:

Proposition 1. *There are $3^{n-3} C_n^3$ minimal directed P-graphs with n vertices labeled $\{1, \ldots, n\}$.*

Proof: The structure of a minimal directed P-graph is given by Theorem 1. The three vertices x, y, and z that form the complete directed subgraph can be chosen in C_n^3 ways. All other vertices u different from x, y and z have $d^-(u) = 0$ and $d^+(u) = 2$ and at u start exactly two edges that end at two out of three vertices from the set $\{x, y, z\}$. We define A_{xy}, A_{xz} and A_{yz} as the sets that consist of the vertices u that connect with either x and y or x and z or y and z. Each vertex u can be in exactly one such set and therefore we have three choices for placing each of the remaining $n-3$ vertices u in these sets. This gives us $3^{n-3}C_n^3$ minimal directed P-graphs with n vertices labeled $\{1, \ldots, n\}$.

We note that among the $3^{n-3}C_n^3$ graphs found above, many are isomorphic. To compute the number of distinct (non-isomorphic) minimal directed P-graphs, we need to introduce a couple of combinatorial results related to integer partitions. We recommend [4] for a very detailed chapter about integer partitions.

Definition 4. *A Partition of a positive integer n in m parts is a m-tuple $(a_1, ..., a_m)$ of integers such that $n = a_1 + a_2 + ... + a_m$, and $a_1 \geq a_2 \geq ... \geq a_m \geq 1$. Let $P(n, m)$ be the number of distinct partitions of n in m parts.*

The following result from [4] will allow us to compute the number of non-isomorphic minimal directed P-graphs:

Proposition 2. *The number $P(n, m)$ also represents the number of partitions of n such that the largest part is equal to m.*

We can now derive the following result:

Proposition 3. *There are $P(n, 3)$ non-isomorphic minimal directed P-graphs.*

Proof: From Theorem 1, we know that the structure of a minimal directed P-graph with n vertices is the following: there are three vertices that form a complete directed subgraph (denote them by x, y and z) and the rest of the vertices have $d^- = 0$ and $d^+ = 2$ and the two edges starting at any of these vertices end in either x, y or z. Hence, we infer that the isomorphism of minimal directed P-graphs depends on the number of vertices connected to either x and y or x and z or y and z. Therefore, the number of non-isomorphic minimal directed P-graphs with n vertices is equal to the number of partitions of $n-3$ in $a_1 + a_2 + a_3$, where $a_1 \geq a_2 \geq a_3 \geq 0$.

We define the function g on the set of partitions with the property above with values in the set of partitions of n given by Definition 4. To each partition of $n-3$ above we associate the partition $n = (a_1 + 1) + (a_2 + 1) + (a_3 + 1)$. One can easily see that $a_1 + 1 \geq a_2 + 1 \geq a_3 + 1 \geq 1$. It is obvious that g is a one-to-one function and therefore there are $P(n, 3)$ non-isomorphic minimal directed P-graphs.

Corollary 1. *The number of non-isomorphic minimal directed P-graphs with n vertices is given by:* $\sum_{i=1}^{\left[\frac{n}{3}\right]} \left(\left[\frac{n-3i}{2}\right] + 1 \right)$.

Proof: From Proposition 3, this number is equal to $P(n, 3)$. From Proposition 2, $P(n,3)$ represents the number of partitions of n with the largest part equal to 3. If in such a partition we have i parts equal to three, then $1 \le i \le \left[\frac{n}{3}\right]$. We are left with $n-3i$ that needs to be partitioned in parts equal with one or two. This can be done in $\left[\frac{n-3i}{2}\right]+1$ ways (we had to add 1 to account for the partition that does not have any parts of size two). Therefore, $P(n,3)=\sum_{i=1}^{\left[\frac{n}{3}\right]}\left(\left[\frac{n-3i}{2}\right]+1\right)$, which is equivalent to the statement of the corollary.

We will conclude the results by giving a density characterization of the directed P_k-graphs.

Definition 5. *Let us denote by $p_k^-(n)$ the number of directed P_k-graphs with n given vertices.*

Theorem 2. *For large values of n, almost all directed graphs with n nodes are directed P_k-graphs or, equivalently, $\frac{p_k^-(n)}{4^{C_n^2}} \to 1$ when $n \to \infty$.*

Proof: According to Definition 3, a graph G is not a directed P_k-graph if there is a k-tuple $(x_{i_1},\dots,x_{i_k})$ of distinct vertices from $V(G)$ that do not have property P_k^-. Let us denote by $G_{i_1 i_2 \dots i_k}$ the set of the graphs for which the k-tuple $(x_{i_1},\dots,x_{i_k})$ of distinct vertices of G does not have property P_k^-. In other words, there is no vertex y of G, different from $x_{i_1},x_{i_2},\dots,x_{i_k}$, such that from each of $x_{i_1},x_{i_2},\dots,x_{i_k}$ there is an edge that ends in y. There are 4^k-2^k ways to connect any of the remaining $n-k$ vertices y with $x_{i_1},x_{i_2},\dots,x_{i_k}$ in such a way that $(x_{i_1},x_{i_2},\dots,x_{i_k})$ does not have Property P_k^- (there are 4^k ways to put edges between y and the k vertices, from which we subtract the number of configurations where all these vertices have an edge ending at y). The k vertices can also be joined among themselves in $4^{C_k^2}$ ways and the remaining $n-k$ vertices can be joined among themselves in $4^{C_{n-k}^2}$ ways. Therefore we infer the following formula:

$$|G_{i_1\dots i_k}| = 4^{C_k^2}(4^k-2^k)^{n-k}4^{C_{n-k}^2}$$

for all $x_{i_1},\dots,x_{i_k}$ distinct vertices.

The set of directed graphs that are not directed P_k-graphs is given by the union of the sets $G_{i_1\dots i_k}$ where $1 \le i_1 < \dots < i_k \le n$. Let us denote this union by U_n. We have:

$$|U_n| \le \sum_{1\le i_1<\dots<i_k\le n} |G_{i_1\dots i_k}| = C_n^k 4^{C_k^2}(4^k-2^k)^{n-k}4^{C_{n-k}^2}$$

By normalizing both sides, we get:

$$\frac{|U_n|}{4^{C_n^2}} \le \frac{C_n^k 4^{C_k^2}(4^k-2^k)^{n-k}4^{C_{n-k}^2}}{4^{C_n^2}} = C_n^k\left(\frac{4^k-2^k}{4^k}\right)^{n-k}.$$

Since C_n^k is a polynomial of degree k in n and since $\frac{4^k-2^k}{4^k} < 1$, from the last equality we infer that $\frac{|U_n|}{4^{C_n^2}} \to 0$ when $n \to \infty$. From the definition of

U_n, $p_k^-(n) = 4^{C_n^2} - |U_n|$. Therefore we have $\frac{p_k^-(n)}{4^{C_n^2}} \to 1$ when $n \to \infty$ or, equivalently, almost all directed graphs with n nodes are directed P_k-graphs for large values of n.

3. **Summary and Future Work**

To conclude, in this paper we presented several results related to directed extensions of P-graphs and P_k-graphs. First, we have identified the structure of minimal directed P-graphs and derived the number of edges in such graphs. Next, the number of non-isomorphic minimal directed P-graphs has been computed via a connection to combinatorial results about integer partitions. Finally, we have proved that directed P_k-graphs make the majority of the directed graphs when the number of vertices becomes large.

A couple of interesting ideas for future work include: extending the notion of Q_k-graphs (as defined in [3]) to directed graphs as well as computing minimal configurations for directed P_k-graphs.

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