

A PROBLEM IN NONLINEAR OPTIMIZATION

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Abstract

The continuous version of the “student optimal control problem” with control constraints is considered and improved theoretical results are provided.

Keywords: *nonlinear optimisation, optimal control*

AMS Classification: 49J21, 49K30, 49N99

1. Introduction

In [4], Ragget, Hempson and Jakes solved the following problem, called “A student optimal control problem”:

Problem 1. *Minimize $\sum_{i=1}^n x_i^2$ subject to $\sum_{i=1}^n x_i = S$, and $x_i \geq 0$ where $S, a_i, b_i, i = 1, 2, \dots, n$ are given positive constants.*

In [3], Muntean I. and Vornicescu N. added constraints $x_i \leq B, i = 1, 2, \dots, n$, with given positive constant B . In [6], Vornicescu N. considered the continuous case namely:

Problem 2. *Minimize the functional*

$$J[u] = \int_0^1 a(t)u^2(t)dt$$

subject to

$$\int_0^1 b(t)u(t)dt = S$$

and control constraints

$$u \in C[0, 1], 0 \leq u(t) \leq B,$$

where $a, b \in C[0, 1]$ are given positive functions, and S and B are given positive constants.

In this paper is considered an extension of the last problem:

Problem 3. Minimize the functional

$$\int_0^1 a(t)u^m(t)dt$$

subject to

$$\int_0^1 b(t)u(t)dt = S$$

and control constraints

$$u \in C[0, 1], 0 \leq u(t) \leq B,$$

where $a, b \in C[0, 1]$ are given positive functions, S and B are given positive constants, and $m > 1$.

2. Preliminaries

A function $u \in C[0, 1]$ is said to be an **admissible strategy** for the Problem 3 if

$$0 \leq u(t) \leq B, \text{ for } t \in [0, 1] \quad (1)$$

and

$$\int_0^1 b(t)u(t)dt = S. \quad (2)$$

For an admissible strategy u , let us denote:

$$J[u] = \int_0^1 a(t)u^m(t)dt.$$

An admissible strategy u^* is said to be an **optimal strategy** for the Problem 3 if for each admissible strategy u we have

$$J[u^*] \leq J[u].$$

An admissible strategy u^* is said to be a **locally optimal strategy** for the Problem 3 if there exists $\delta > 0$ such that, if u is an admissible strategy verifying

$$\max\{|u^*(t) - u(t)|, t \in [0, 1]\} < \delta$$

then

$$J[u^*] \leq J[u].$$

We need to consider four cases, discussed in Propositions 1, 2, 3 and Theorem 1.

Proposition 1. *If*

$$\frac{S}{B} > \int_0^1 b(t)dt,$$

then the Problem 3 has no solution.

Indeed,

$$\int_0^1 b(t)u(t)dt \leq B \int_0^1 b(t)dt < S$$

for each function u verifying condition (1).

Proposition 2. *If*

$$\frac{S}{B} = \int_0^1 b(t)dt,$$

then the unique admissible strategy is $u = B$.

The proof is straightforward.

Proposition 3. *If*

$$\frac{S}{B} \leq \min\{a^{\frac{1}{m-1}}(s)b^{\frac{-1}{m-1}}(s), s \in [0, 1]\} \cdot \int_0^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt$$

then the optimal strategy is

$$u^*(t) = Sa^{\frac{-1}{m-1}}(t)b^{\frac{1}{m-1}}(t) \left(\int_0^1 a^{\frac{-1}{m-1}}(s)b^{\frac{m}{m-1}}(s)ds \right)^{-1}$$

and

$$J[u^*] = S^m \left(\int_0^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt \right)^{-m+1}. \quad (3)$$

Proof: An easy computation show that u^* is an admissible strategy and satisfies (3). In order to prove that u^* is an optimal strategy we will consider an arbitrary admissible strategy u . Using Hölder's inequality with $p = m$ and $q = \frac{m}{m-1}$ we obtain that:

$$\begin{aligned} S &= \int_0^1 b(t)dt = \int_0^1 a^{\frac{1}{m}}(t)u(t) \cdot a^{\frac{-1}{m}}(t)b(t)dt \leq \\ &\leq \left(\int_0^1 a(t)u^m(t)dt \right)^{\frac{1}{m}} \left(\int_0^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt \right)^{\frac{m-1}{m}} \end{aligned}$$

whence

$$J[u] \geq S^m \left(\int_0^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt \right)^{-m+1} = J[u^*].$$

It remains to study the case when

$$\min\{a^{\frac{1}{m-1}}(s)b^{-\frac{1}{m-1}}(s), s \in [0, 1]\} \cdot \int_0^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt < \frac{S}{B} < \int_0^1 b(t)dt \quad (4)$$

which is in fact the main case.

3. Main result

For the case (4) we will give a characterization of a locally optimal strategy. The idea of the proof is to consider the variation of the functional J .

Before giving the main result (Theorem 1), we need to prove the following lemma.

Lemma 1. *If \bar{u} is a locally optimal strategy for Problem 3 and if there exist $t_1, t_2 \in]0, 1[$ such that*

$$a(t_1)b^{-1}(t_1)\bar{u}^{m-1}(t_1) < a(t_2)b^{-1}(t_2)\bar{u}^{m-1}(t_2) \quad (5)$$

then

$$\bar{u}(t_1) = B.$$

Proof: Suppose, contrary to our claim, that

$$\bar{u}(t_1) < B.$$

There exists $\alpha > 0$ such that

$$[t_1 - \alpha, t_1 + \alpha] \subset [0, 1], [t_2 - \alpha, t_2 + \alpha] \subset [0, 1] \quad (6)$$

$$[t_1 - \alpha, t_1 + \alpha] \cap [t_2 - \alpha, t_2 + \alpha] = \emptyset \quad (7)$$

$$a(c_1)b^{-1}(c_2)\bar{u}^{m-1}(c_1) < a(c_3)b^{-1}(c_4)\bar{u}^{m-1}(c_3) \quad (8)$$

for all $c_1, c_2 \in [t_1 - \alpha, t_1 + \alpha]$, $c_3, c_4 \in [t_2 - \alpha, t_2 + \alpha]$.

Let us denote

$$P = \int_{t_1-\alpha}^{t_1+\alpha} b(t)[\alpha^2 - (t - t_1)^2]dt$$

and

$$Q = \int_{t_2-\alpha}^{t_2+\alpha} b(t)[\alpha^2 - (t - t_2)]dt.$$

By applying the mean value theorem, we obtain that there exist

$$c_2 \in [t_1 - \alpha, t_1 + \alpha], c_4 \in [t_2 - \alpha, t_2 + \alpha]$$

such that

$$P = b(c_2) \int_{t_1-\alpha}^{t_1+\alpha} [\alpha^2 - (t - t_1)^2] dt = \frac{4}{3} \alpha^3 b(c_2), Q = \frac{4}{3} \alpha^3 b(c_4). \quad (9)$$

Let $\varepsilon_0 > 0$ be sufficiently small such that for $0 < \varepsilon \leq \varepsilon_0$ we have:

$$\bar{u}(t) + \frac{\varepsilon}{P} [\alpha^2 - (t - t_1)^2] < B, \text{ if } t \in [t_1 - \alpha, t_1 + \alpha],$$

$$\bar{u}(t) - \frac{\varepsilon}{Q} [\alpha^2 - (t - t_2)^2] > 0, \text{ if } t \in [t_2 - \alpha, t_2 + \alpha].$$

It is easy to see that for $\varepsilon \in [0, \varepsilon_0]$, the function $u_\varepsilon : [0, 1] \rightarrow [0, 1]$ defined by:

$$u_\varepsilon(t) = \begin{cases} \bar{u}(t), & t \notin [t_1 - \alpha, t_1 + \alpha] \cup [t_2 - \alpha, t_2 + \alpha] \\ \bar{u}(t) + \frac{\varepsilon}{P} [\alpha^2 - (t - t_1)^2], & t \in [t_1 - \alpha, t_1 + \alpha] \\ \bar{u}(t) - \frac{\varepsilon}{Q} [\alpha^2 - (t - t_2)^2], & t \in [t_2 - \alpha, t_2 + \alpha] \end{cases}$$

is an admissible strategy for the Problem 3.

Now we define the function

$$L : [0, \varepsilon_0] \rightarrow R, L(\varepsilon) = \int_0^1 a(t) \bar{u}_\varepsilon^m(t) dt.$$

If $t_1 < t_2$ then:

$$\begin{aligned} L(\varepsilon) &= \int_0^{t_1-\alpha} a(t) \bar{u}^m(t) dt + \int_{t_1-\alpha}^{t_1+\alpha} a(t) \left[\bar{u}(t) + \frac{\varepsilon}{P} [\alpha^2 - (t - t_1)^2] \right]^m dt + \\ &+ \int_{t_1+\alpha}^{t_2-\alpha} a(t) \bar{u}^m(t) dt + \int_{t_2-\alpha}^{t_2+\alpha} a(t) \left[\bar{u}(t) - \frac{\varepsilon}{Q} [\alpha^2 - (t - t_2)^2] \right]^m dt + \\ &+ \int_{t_2+\alpha}^1 a(t) \bar{u}^m(t) dt \end{aligned}$$

and

$$\begin{aligned} L'(\varepsilon) &= \frac{m}{P} \int_{t_1-\alpha}^{t_1+\alpha} a(t) \left[\bar{u}(t) + \frac{\varepsilon}{P} [\alpha^2 - (t - t_1)^2] \right]^{m-1} [\alpha^2 - (t - t_1)^2] dt - \\ &- \frac{m}{Q} \int_{t_2-\alpha}^{t_2+\alpha} a(t) \left[\bar{u}(t) - \frac{\varepsilon}{Q} [\alpha^2 - (t - t_2)^2] \right]^{m-1} [\alpha^2 - (t - t_2)^2] dt. \end{aligned}$$

If $t_1 > t_2$ then $L'(\varepsilon)$ is given by the same formula.

Taking into account (9) we obtain:

$$L'(0_+) = \frac{3m}{4\alpha^3} \left\{ \frac{1}{b(c_2)} \int_{t_1-\alpha}^{t_1+\alpha} a(t)\bar{u}^{m-1}(t)[\alpha^2 - (t-t_2)^2]dt - \frac{1}{b(c_4)} \int_{t_2-\alpha}^{t_2+\alpha} a(t)\bar{u}^{m-1}(t)[\alpha^2 - (t-t_2)^2]dt \right\}.$$

By applying again the mean value theorem, we obtain that there exist

$$c_1 \in [t_1 - \alpha, t_1 + \alpha], c_3 \in [t_2 - \alpha, t_2 + \alpha]$$

such that:

$$L'(0_+) = m[b^{-1}(c_2)a(c_1)\bar{u}^{m-1}(c_1) - b^{-1}(c_4)a(c_3)\bar{u}^{m-1}(c_3)] < 0.$$

Hence there exists $\varepsilon \in]0, \varepsilon_0[$ such that $L(\varepsilon) < L(0)$ or $J[u_\varepsilon] < J[\bar{u}]$ which contradicts the optimality of \bar{u} .

Theorem 1. *If the inequalities in 4 are verified and if \bar{u} is a locally optimal strategy for Problem 3, then there exists $t_0 \in [0, 1]$ such that*

$$\bar{u}(t) = \begin{cases} B & a(t)b^{-1}(t) < a(t_0)b^{-1}(t_0) \\ Ba^{\frac{1}{m-1}}(t_0)b^{-\frac{1}{m-1}}(t_0)a^{-\frac{1}{m-1}}(t)b^{\frac{1}{m-1}}(t) & a(t)b^{-1}(t) \geq a(t_0)b^{-1}(t_0) \end{cases}$$

Proof: Let us denote $C = \max\{a(t)b^{-1}(t)\bar{u}^{m-1}(t), t \in [0, 1]\}$, $E_1 = \{t \in [0, 1], a(t)b^{-1}(t)\bar{u}^{m-1}(t) = C\}$ and $E_2 = [0, 1] \setminus E_1$.

Firstly, we will show that $E_2 \neq \emptyset$. Supposing the contrary, we have

$$\bar{u} = C^{\frac{1}{m-1}} a^{-\frac{1}{m-1}}(t) b^{\frac{1}{m-1}}(t),$$

for every $t \in [0, 1]$. From (2) we obtain:

$$S = \int_0^1 b(t)\bar{u}(t)dt = C^{\frac{1}{m-1}} \int_0^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt$$

whence

$$C^{\frac{1}{m-1}} = S \left(\int_0^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt \right)^{-1}$$

and

$$\bar{u}(t) = Sa^{-\frac{1}{m-1}}(t)b^{\frac{1}{m-1}}(t) \left(\int_0^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt \right)^{-1}.$$

If

$$a^{\frac{1}{m-1}}(t_1)b^{-\frac{1}{m-1}}(t_1) = \min\{a^{\frac{1}{m-1}}(t)b^{-\frac{1}{m-1}}(t), t \in [0, 1]\}$$

then from (4) we obtain:

$$\bar{u}(t_1) = Sa^{-\frac{1}{m-1}}(t_1)b^{\frac{1}{m-1}}(t_1) \left(\int_0^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt \right)^{-1} >$$

$$B \min\{a^{\frac{1}{m-1}}(t)b^{-\frac{1}{m-1}}(t), t \in [0, 1]\} \cdot a^{-\frac{1}{m-1}}(t_1)b^{\frac{1}{m-1}}(t_1) = B$$

which contradicts (1). In conclusion, E_2 is a nonempty open set in the induced topology on interval $[0, 1]$.

Let $]t_0, t_1[$ (or $]t_1, t_0[$) be a maximal open interval included in E_2 , such that $t_0 \in E_1$. If $t \in E_2$ then

$$a(t)b^{-1}(t)\bar{u}^{m-1}(t) < C = a(t_0)b^{-1}(t_0)\bar{u}^{m-1}(t_0)$$

and from Lemma 1 it follows $\bar{u}(t) = B$.

From continuity of the function \bar{u} it results that $\bar{u}(t_0) = B$, whence $a(t)b^{-1}(t) < a(t_0)b^{-1}(t_0)$.

If $t \in E_1$ then $a(t)b^{-1}(t)\bar{u}^{m-1}(t) = C = a(t_0)b^{-1}(t_0)B^{m-1}$.

Hence $a(t)b^{-1}(t) \geq a(t_0)b^{-1}(t_0)$ and

$$\bar{u}(t) = Ba^{\frac{1}{m-1}}(t_0)b^{-\frac{1}{m-1}}(t_0)a^{\frac{-1}{m-1}}(t)b^{\frac{1}{m-1}}(t).$$

Corollary 1. *If ab^{-1} is a monotone function then for Problem 3 there exists a unique admissible strategy verifying (9).*

Proof:

i) Suppose that ab^{-1} is an increasing function. We remark that in this case $0 \leq t < t_0$ is equivalent with $a(t)b^{-1}(t) < a(t_0)b^{-1}(t_0)$

Let us consider the function $F : [0, 1] \rightarrow R$ defined by:

$$F(s) = B \int_0^s b(t)dt + Ba^{\frac{1}{m-1}}(s)b^{-\frac{1}{m-1}}(s) \int_s^1 a^{-\frac{1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt.$$

We will show that F is an increasing function. Indeed, if $0 \leq s_1 < s_2 \leq 1$ then

$$\begin{aligned} F(s_2) - F(s_1) &= B \int_{s_1}^{s_2} b(t) \left[1 - a^{\frac{1}{m-1}}(s_1)b^{-\frac{1}{m-1}}(s_1)a^{\frac{-1}{m-1}}(t)b^{\frac{1}{m-1}}(t) \right] dt + \\ &+ B \left[a^{\frac{1}{m-1}}(s_2)b^{-\frac{1}{m-1}}(s_2) - a^{\frac{1}{m-1}}(s_1)b^{-\frac{1}{m-1}}(s_1) \right] \int_{s_2}^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt > 0. \end{aligned}$$

From (4) it results

$$F(0) = Ba^{\frac{1}{m-1}}(0)b^{-\frac{1}{m-1}}(0) \int_0^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt < S$$

and

$$F(1) = B \int_0^1 b(t)dt > S.$$

Hence there exists a unique $t_0 \in]0, 1[$ such that $F(t_0) = S$.

If we consider the function $u^* : [0, 1] \rightarrow R$ defined by

$$u^*(t) = \begin{cases} B & \text{if } 0 \leq t < t_0 \\ Ba^{\frac{1}{m-1}}(t_0)b^{-\frac{1}{m-1}}(t_0)a^{\frac{-1}{m-1}}(t)b^{\frac{1}{m-1}}(t) & \text{if } t_0 \leq t \leq 1 \end{cases}$$

then $0 < u^*(t) \leq B$ for $t \in [0, 1]$ and $J[u^*] = F(t_0) = S$.

ii) Suppose that ab^{-1} is a decreasing function. In this case $0 \leq t < t_0$ is equivalent with $a(t)b^{-1}(t) \geq a(t_0)b^{-1}(t_0)$.

We consider the function $G : [0, 1] \rightarrow R$ defined by

$$G(s) = Ba^{\frac{1}{m-1}}(s)b^{-\frac{1}{m-1}}(s) \int_0^s a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt + B \int_s^1 b(t)dt$$

We have that G is a decreasing function since

$$\begin{aligned} G(s_2) - G(s_1) &= b[a^{\frac{1}{m-1}}(s_2)b^{-\frac{1}{m-1}}(s_2) - \\ &\quad - a^{\frac{1}{m-1}}(s_1)b^{-\frac{1}{m-1}}(s_1)] \int_0^{s_1} a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt + \\ &\quad + B \int_{s_1}^{s_2} b(t)[a^{\frac{1}{m-1}}(s_2)b^{-\frac{1}{m-1}}(s_2)a^{\frac{-1}{m-1}}(t)b^{\frac{1}{m-1}}(t) - 1]dt < 0 \end{aligned}$$

if $0 \leq s_1 < s_2 \leq 1$.

Since

$$G(0) = B \int_0^1 b(t)dt > S$$

and

$$G(1) = Ba^{\frac{1}{m-1}}(1)b^{-\frac{1}{m-1}}(1) \int_0^1 a^{\frac{-1}{m-1}}(t)b^{\frac{m}{m-1}}(t)dt < S,$$

we obtain that there exists a unique $t_0 \in]0, 1[$ such that $G(t_0) = S = J[u^*]$, where

$$u^*(t) = \begin{cases} Ba^{\frac{1}{m-1}}(t_0)b^{-\frac{1}{m-1}}(t_0)a^{\frac{-1}{m-1}}(t)b^{\frac{1}{m-1}}(t) & \text{if } 0 \leq t < t_0, \\ B & \text{if } t_0 \leq t \leq 1. \end{cases}$$

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