SOME ESTIMATIONS OF THE MANDELBROT SET

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Abstract

The aim of this article is to give some estimations of the Mandelbrot set. We also compute how big is the radius of a ball contained entirely into the Mandelbrot set.

Keywords: *Julia sets, the Mandelbrot set*

AMS Classification: 37F45

1. **Julia sets**

Julia sets arise in connection with the iteration of a function of a complex variable, so are related to the dynamical systems. In general a Julia set is a dynamical repeller. However, by specializing to functions that are analytic on the complex plane (such as polynomials), we can use the techniques of complex variable theory to obtain more detailed information about the structure of such sets. The Julia set is named after the French mathematician Gaston Julia who investigated their properties circa 1915 and culminated in his famous paper in 1918.

We take $f : \mathbb{C} \to \mathbb{C}$ to be a polynomial of degree n with complex coefficients, $f(z) = a_n z^n + ... + a_0$. We write f^k for the k-fold composition $f \circ f \circ ... \circ f$ of the function f, so that $f^k(w)$ is the kth iterate $f(f...f(w)...))$ of w. Julia sets are defined in terms of the behaviour of the iterates $f^k(z)$ for large k.

First, we define the filled-in Julia set of the polynomial f.

Definition 1. The *filled-in Julia set* of a f is

$$K(f) = \{z \in \mathbb{C} \mid f^k(z) \overset{k\to\infty}{\nrightarrow} \infty\}$$

Definition 2. The *Julia set* of f is the boundary of the filled-in Julia set

$$J(f) = \partial K(f)$$

Thus $z \in J(f)$ if in every neighbourhood of z there are points w and v with $f^k(w) \to \infty$ and $f^k(v) \nrightarrow \infty$.

Definition 3. The complement of the Julia set is called the *Fatou set* or *stable set* $F(f)$.

We consider the case of quadratic functions f on $\mathbb{C}$. This study can be reduced to the study of the polynomials of the form $f_c(z) = z^2 + c$, because every quadratic function is similar to an $f_c(z)$ for some $c \in \mathbb{C}$, that means there exists a similarity h of the plane which transforms the dynamical picture of f to that of f_c. In particular, the Julia set of f is the image under h^{-1} of the Julia set of f_c.

Indeed, if $h(z) = \alpha z + \beta$, $\alpha \neq 0$, then $h^{-1}(f_c(h(z))) = (\alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c - \beta)/\alpha$. By choosing appropriate values of α, β and c we can make this expression into any quadratic function that we want.

We have one first result, which was proven in [1], about the Julia set of a quadratic form.

Theorem 1. [1] *If* $|c| < \frac{1}{4}$ *then* $J(f_c)$ *is a simple closed curve.*

Concerning the topological properties of the filled-in Julia and Julia sets of a general polynomial we have the following result:

Theorem 2. [2] *Let f be a polynomial of degree $n \geq 2$ with complex coefficients. Then the filled-in Julia set $K(f)$ and the Julia set $J(f)$ have the following properties*:

1) *$K(f)$ and $J(f)$ are non-empty and compact with $J(f) \subset K(f)$*;
2) *$J(f)$ has an empty interior*;
3) *$J(f) = J(f^p)$ for every positive integer p*;
4) *$J(f)$ is a perfect set.*

We have the following result concerning the shape of a Julia set.

Theorem 3. *Let $f_c(z) = z^2 + c$ and $J(f_c)$ be its Julia set. Then the Julia set $J(f_c)$ is symmetric about the origin.*

Proof: We have that $f_c(z) = f_c(-z)$, $\forall z \in \mathbb{C}$, so $f_c^k(z) = f_c^k(-z)$ for all $k \in \mathbb{N}^*$. Thus $f_c^k(-z) \to \infty$ if and only if $f_c^k(z) \to \infty$, so $-z \in K(f_c)$ if and only if $z \in K(f_c)$. Thus the filled-in Julia set $K(f_c)$ is symmetric about the origin, and its boundary, the Julia set $J(f_c)$, is also symmetric about the origin.

2. **The Mandelbrot set**

The Mandelbrot set is probably one of the most well known fractals, and probably one of the most widely implemented fractal in fractal plotting programs. It was originally discovered by Benoit B. Mandelbrot, hence the name.

Definition 4. We define the *Mandelbrot set* M to be the set of parameters $c \in \mathbb{C}$ for which the Julia set of $f_c(z) = z^2 + c$ is connected, that means

$$M = \{c \in \mathbb{C} \mid J(f_c) \text{ is connected}\}$$

In [1] it is given a characterization of the Mandelbrot set, which is more useful in pratice and which we will use during this paper.

Theorem 4. [1] *If M is the Mandelbrot set then M satisfies*

$$M = \{c \in \mathbb{C} \mid \{f_c^k(0)\}_{k\geq 1} \textit{ is bounded}\} = \{c \in \mathbb{C} \mid \{f_c^k(0)\}_k \overset{k\to\infty}{\nrightarrow} \infty\} \qquad (1)$$

Concerning the topological properties of the Mandelbrot set we have the following result:

Theorem 5. [3],[4] *Let M be the Mandelbrot set. Then the followings are true*:

a) M is compact,
b) Its complement $\mathbb{C} \setminus M$ is connected,
c) Each connected component of its interior is simply-connected,
d) M is connected.

We give now some estimations of the Mandelbrot set.
We denote by $B(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$, with $a \in \mathbb{C}$ and $r > 0$.

Theorem 6. *Let $f_c(z) = z^2 + c$ and M be the Mandelbrot set. Then the folowings are true:*

a) *If $c > \frac{1}{4}$ then $c \notin M$;*
b) $B(0, \frac{1}{4}) \subset M$, $B(-1, \frac{1}{15}) \subset M$ *and* $B(-\frac{1}{4}, \frac{1}{28}) \subset M$;
c) *If* $\mid c \mid > 2$ *then* $c \notin M$;
d) $-2 \in M$.

Proof:
a) We consider $c \in \mathbb{R}, c > \frac{1}{4}$ and $z \in \mathbb{R}$. Then

$$f_c(z) - z = z^2 + c - z = (z - \tfrac{1}{2})^2 + (c - \tfrac{1}{4}) \geq c - \tfrac{1}{4}, \text{ because } z \text{ is real.}$$

Thus $f_c(z) \geq z + (c - \frac{1}{4})$, $\forall z \in \mathbb{R}$.

We observe that $f_c(z)$ is real when z is real, so it follows that

$$f_c^k(z) \geq z + k(c - \tfrac{1}{4}) \overset{k\to\infty}{\to} \infty \Rightarrow z \notin J(f_c), \forall z \in \mathbb{R}. \qquad (2)$$

But $J(f_c)$ is non-empty and from theorem 3. it is symmetric $\Rightarrow \exists\ z_0 \in \mathbb{C}\backslash\{0\}$ such that $z_0, -z_0 \in J(f_c)$. Because $J(f_c)$ does not contain the real axis, we have that z_0 and $-z_0$ lie on opposite sides of the real axis, so $J(f_c)$ cannot be connected. Thus $c \notin M$.

b) Suppose that $\mid c \mid \leq \frac{1}{4}$. For $|z| \leq \frac{1}{2}$, using the triangle inequality we have that

$$|f_c(z)| = |z^2 + c| \leq |z|^2 + |c| \leq (\tfrac{1}{2})^2 + \tfrac{1}{4} = \tfrac{1}{2}. \qquad (3)$$

Applying this inductively we get $|f_c^k(0)| \leq \frac{1}{2}$, for all $k \in \mathbb{N}$.
Thus if $\mid c \mid \leq \frac{1}{4}$, $f_c^k(0) \nrightarrow \infty$, so by theorem 4., $c \in M$. So $B(0, \frac{1}{4}) \subset M$.

Suppose that $\mid c + 1 \mid \leq \frac{1}{15} \Rightarrow |c| \leq \frac{16}{15}$. For $|z| \leq \frac{1}{10}$ we have that

$$\begin{gathered} |f_c(f_c(z))| = |(z^2 + c)^2 + c| = |z^4 + 2cz^2 + c^2 + c| \leq \\ |z|^4 + 2|c||z|^2 + |c^2 + c| = |z|^4 + 2|c||z|^2 + |c||c+1| \leq \\ (\tfrac{1}{10})^4 + 2 \cdot \tfrac{16}{15} \cdot (\tfrac{1}{10})^2 + \tfrac{16}{15} \cdot \tfrac{1}{15} = \tfrac{41645}{450000} < \tfrac{1}{10}. \qquad (4) \end{gathered}$$

Thus for $|c+1| \leq \frac{1}{15}$, applying this inductively, $|f_c^{2k}(0)| < \frac{1}{10}$ for all $k \in \mathbb{N}$. Hence $f_c^{2k}(0) \nrightarrow \infty$, so by theorem 4., $c \in M \Rightarrow B(-1, \frac{1}{15}) \subset M$.

Suppose that $\mid c + \frac{1}{4} \mid \leq \frac{1}{28} \Rightarrow |c| \leq \frac{1}{28} + \frac{1}{4} = \frac{8}{28}$ and $|c+1| \leq \frac{1}{28} + \frac{3}{4} = \frac{22}{28}$. For $|z| \leq \frac{2}{7}$ we have that

$$\begin{gathered} |f_c(f_c(z))| = |(z^2 + c)^2 + c| = |z^4 + 2cz^2 + c^2 + c| \leq \\ |z|^4 + 2|c||z|^2 + |c^2 + c| = |z|^4 + 2|c||z|^2 + |c||c+1| \leq \\ (\tfrac{2}{7})^4 + 2 \cdot \tfrac{8}{28} \cdot (\tfrac{2}{7})^2 + \tfrac{8}{28} \cdot \tfrac{22}{28} = \tfrac{18676}{67228} < \tfrac{2}{7}. \qquad (5) \end{gathered}$$

Thus for $|c + \frac{1}{4}| \leq \frac{1}{28}$, applying this inductively, $|f_c^{2k}(0)| < \frac{2}{7}$ for all $k \in \mathbb{N}$. Hence $f_c^{2k}(0) \nrightarrow \infty$, so by theorem 4., $c \in M \Rightarrow B(-\frac{1}{4}, \frac{1}{28}) \subset M$.

c) If $\mid c \mid > 2$ we may choose $\varepsilon > 0$ such that $|c| > 2 + \varepsilon$ and for $|z| \geq |c|$ we have

$$\begin{gathered} |f_c(z)| = |z^2 + c| \geq |z|^2 - |c| = |z|(|z| - \tfrac{|c|}{|z|}) \geq |z|(|c| - 1) \geq \\ |z|(2 + \varepsilon - 1) = |z|(1 + \varepsilon). \qquad (6) \end{gathered}$$

Because $f_c(0) = c \Rightarrow |f_c(f_c(0))| \geq |f_c(0)|(1+\varepsilon) = |c|(1+\varepsilon)$.
Applying this inductively we obtain that $|f_c^k(0)| \geq (1+\varepsilon)^k |c| \to \infty$. Thus if $\mid c \mid > 2$, by theorem 4., $c \notin M$.

d) We observe that $f_{-2}(0) = -2$ and $f_{-2}^2(0) = f_{-2}(-2) = 2$. Since 2 is a fixed point of f ($f_{-2}(2) = 2$), it folows that $f_{-2}^k(0) = 2$ for $k = 2, 3, ...$, so $f_{-2}^k(0) \nrightarrow \infty$. Thus by theorem 4., $-2 \in M$.

Now we will investigate how big could be the radius of a ball contained entirely into the Mandelbrot set. First we make a remark.

Remark 1. If we construct some conditions in which the k-th iterate of the function f_c remains bounded as k goes to infinity, then c belongs to the Mandelbrot set.

Theorem 7. *Let $f_c(z) = z^2 + c$. We fix a point $a \in \mathbb{C}$ and we consider $c \in \mathbb{C}$ such that $|c - a| \le r(a)$, where $r(a)$ is a positive number which depends on a. We suppose that there exists $R > 1$ such that $|z| \le \frac{1}{R} \Rightarrow |f_c(z)| \le \frac{1}{R}$.*
Then $r(a) > |a| - 2$.

Proof: We consider $z \in \mathbb{C}$ with $|z| \le \frac{1}{R}$, $R > 1$. By hypothesis we have that $|f_c(z)| \le \frac{1}{R}$. By remark 1. this assures that c belongs to M. But, using the triangle inequality we have

$$|c| - \tfrac{1}{R^2} \le |c| - |z^2| \le |c + z^2| = |f_c(z)| \le \tfrac{1}{R}. \qquad (7)$$

Thus $|c| \le \frac{1}{R^2} + \frac{1}{R} = \frac{R+1}{R^2}$.
We also have the condition that $|c - a| \le r(a)$, which implies

$$|a| - |c| \le |c - a| \le r(a) \Rightarrow |c| \ge |a| - r(a). \qquad (8)$$

Hence $|a| - r(a) \le \frac{R+1}{R^2} \Rightarrow r(a) \ge |a| - \frac{R+1}{R^2}$. (9)
If we consider now the function $g : (1, \infty) \to (0, \infty)$, $g(x) = \frac{x+1}{x^2}$, we get that g is strictly decreasing, hence $g(x) \le \lim_{x \to 1} g(x) = 2$.
Thus $r(a) \ge |a| - \frac{R+1}{R^2} > |a| - 2$.

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