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Abstract

We propose some necessary and/or sufficient conditions with respect to scalarization for ε -efficiency, ε -weak efficiency and ε -proper efficiency portfolios for a reinsurance market in nonconvex optimization problem.

Keywords: *reinsurance market, portfolio, efficiency*

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1. Introduction

The reinsurance problem appears at first sight to be a problem which can be analyzed in terms of classical economic theory, if the objectives of the companies have been formulated in an operational manner by the help of Bernoulli's utility concept: it must not be maximized the expected gain, but the expected utility of the gain [3]. However, closer investigations show that the economic theory is only relevant part of the way. Then the problem becomes a problem of cooperation between parties who have conflicting interests, and who are free to form and break any coalitions which may serve their particular interests [12]. Classical economic theory is powerless when it comes to analyze such problems. In the last decades it is shown that there are many possibilities to study and to explain the apparently chaotic situation by helping the games theory or convex analysis.

In this paper we present some new concepts who generalize and refine some result about efficient portfolio, weakly efficient portfolio and properly efficient portfolio based on paper by Ghica[2009] and extend results from convex analysis established by Li and Wang [1998], Kaliszewski [1995]. A basic result of convex analysis is the Fundamental Theorem on Convex Functions [4], [13] which states that an efficient solution of a convex vector optimization problem necessarily minimizes a linear combination of objectives functions. Later in 1995, Kaliszewski characterize efficient solutions without assuming convexity conditions. All these results help us to establish some new concept for the portfolios in a reinsurance market in nonconvex optimization problem.

If we see N as a group of n reinsurers, having preferences $\geq_i, i \in N$, over a suitable set of random variables denoted by R , or gambles with realizations

(outcomes) in some $A \subseteq R$, we represent these preferences by von Neumann-Morgenstern expected utility, meaning that there is a set of continuous utility functions $u_i : R \rightarrow \mathbb{R}$, such that $X \geq_i Y$ if and only if $Eu_i(X) \geq Eu_i(Y)$, where by the symbol E we denoted the mean operator. We assume monotonic preferences, and risk aversion, so that, we have $u_i'(w) > 0$, $u_i''(w) \leq 0$ for all w in the relevant domains [5]. In some of the cases we shall also require strict risk aversion, meaning strict concavity for some u_i . For a better understanding we presume that each agent is endowed with a random variable payoff X_i called initial portfolio. More precisely, there exists a probability space (Ω, \mathcal{K}, P) such that we have the payoff $X_i(\omega)$ when $\omega \in \Omega$ occurs and, more, we have the both expected values and variances exist for all these initial portfolios, which means that all $X_i \in L^2(\Omega, \mathcal{K}, P)$ [6]. Because every agent can negotiate any affordable contracts then we will have a new set of random variables Y_i , $i \in N$, representing the final portfolios. We say if the following condition exists $\sum_{i=1}^n Z_i \leq \sum_{i=1}^n X_i = X_N$ then an allocation $Z = (Z_1, Z_2, \dots, Z_n)$ is called feasible [1].

The paper is organized as follows : In Section 2 we define and characterize concepts like ε -efficient, ε -weakly efficient, ε -properly efficient portfolios, in Sections 3 and 4 we give some sufficient and/or necessary conditions for ε -efficient portfolios and for ε -weak efficient portfolios in a reinsurance market in nonconvex optimization problem with respect to scalarization.

2. ε -Efficient, ε -weak efficient, and ε -proper efficient portfolios

Consider the following optimization problem:

$$\min (Eu_1(Z_1), Eu_2(Z_2), \dots, Eu_n(Z_n)). \quad (2.1)$$

$$Z \in \mathcal{Z}$$

In this paper the feasible set of portfolios \mathcal{Z} in (2.1) will be given in the following form: $\mathcal{Z} = \left\{ Z \mid \sum_{i \in N} Z_i \leq \sum_{i \in N} X_i \right\}$ where (X_1, \dots, X_n) is the initial portfolio X . Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \mathbb{R}^n$ be a tolerance vector satisfying $\varepsilon_i \geq 0$, $i = \overline{1, n}$.

Definition 1. $Y \in \mathcal{Z}$ is called an ε -weakly efficient portfolio of (2.1) if there exists no $Z \in \mathcal{Z}$ such that

$$Eu_i(Z_i) < Eu_i(Y_i) - \varepsilon_i, \quad i = 1, \dots, n$$

Definition 2. $Y \in \mathcal{Z}$ is called an ε -efficient portfolio of (2.1) if there exist no $Z \in \mathcal{Z}$ such that

$$Eu_i(Z_i) \leq Eu_i(Y_i) - \varepsilon_i, \quad i = 1, \dots, n$$

with at least one strict inequality.

Definition 3. $Y \in \mathcal{Z}$ is called an ε -properly efficient portfolio of (2.1) if there exists $M > 0$ such that for any $i \in N$ and $Z \in \mathcal{Z}$ such that

$$Eu_i(Z_i) < Eu_i(Y_i) - \varepsilon_i, \quad i = 1, \dots, n$$

there exists $j \in N$ with

$$Eu_j(Z_j) > Eu_j(Y_j) - \varepsilon_j$$

and

$$[Eu_i(Y_i) - \varepsilon_i - Eu_i(Z_i)] / [Eu_j(Z_j) - Eu_j(Y_j) + \varepsilon_j] \leq M.$$

Definition 4. Let be $\varphi : \mathcal{Z} \rightarrow \mathbb{R}$ and e a positive number. Then $Y \in \mathcal{Z}$ is called e -optimal portfolio of the optimization problem

$$\min \varphi(Z), \quad Z \in \mathcal{Z}$$

if $\varphi(Z) \geq \varphi(Y) - e$ for any $Z \in \mathcal{Z}$.

Remark 1. ε -proper efficient portfolio \Rightarrow ε -efficient portfolio \Rightarrow ε -weak efficient portfolio.

Remark 2. For $\varepsilon = 0$, an ε (properly, weakly) efficient portfolio of (2.1) is a (properly, weakly) efficient portfolio.

Theorem 1. $Y \in \mathcal{Z}$ is an ε -weakly efficient portfolio of (2.1) if and only if for any $y_i^* < Eu_i(Y_i)$, $i = \overline{1, n}$, Y is $\max_{i=\overline{1, n}} \lambda_i \varepsilon_i$ -optimal to the following optimization problem

$$\min \max_{i=\overline{1, n}} \lambda_i [Eu_i(Z_i) - y_i^*] \quad \text{subject to } Z \in \mathcal{Z}, \quad (2.2)$$

where $\lambda_i = |Eu_i(Y_i) - y_i^*|^{-1}$, $i = \overline{1, n}$.

Proof: Suppose that $Y \in \mathcal{Z}$ is an ε -weakly efficient portfolio of (2.1). We prove that Y is $\max_{i=\overline{1, n}} \lambda_i \varepsilon_i$ -optimal to (2.2). By definition of ε -weakly efficient portfolio we have not solution $Z \in \mathcal{Z}$ for the following system:

$$Eu_i(Z_i) < Eu_i(Y_i) - \varepsilon_i, \quad i = 1, \dots, n.$$

By first theorem, for any $\delta_1, \delta_2, \dots, \delta_n$ negative real numbers there exist positive numbers $\lambda_i = -\delta_i^{-1}$, $i = \overline{1, n}$, such that

$$\max_{i=\overline{1, n}} \lambda_i [Eu_i(Z_i) - Eu_i(Y_i) + \varepsilon_i - \delta_i] \geq 1, \quad \forall Z \in \mathcal{Z}.$$

Let $y_i^* < Eu_i(Y_i)$, $\delta_i = y_i^* - Eu_i(Y_i)$ and $\lambda_i = -\delta_i^{-1} > 0$, $i = \overline{1, n}$ and

$$\max_{i=\overline{1, n}} \lambda_i [Eu_i(Z_i) - y_i^*] \geq 1, \quad \forall Z \in \mathcal{Z}.$$

Because $\max_{i=1,n} \lambda_i [Eu_i(Z_i) - y_i^* + \varepsilon_i] \leq \max_{i=1,n} \lambda_i [Eu_i(Z_i) - y_i^*] + \max_{i=1,n} \lambda_i \varepsilon_i$, $\forall Z \in \mathcal{Z}$. Applying the definition of the ε -optimal portfolio we just prove that Y is a $\max_{i=1,n} \lambda_i \varepsilon_i$ -optimal solution of (2.2).

Conversely, we suppose now that Y is a $\max_{i=1,n} \lambda_i \varepsilon_i$ -optimal solution of (2.2) and want to prove that $Y \in \mathcal{Z}$ is an ε -weakly efficient portfolio of (2.1). Assume that $Y \in \mathcal{Z}$ was not an ε -weakly efficient portfolio of (2.1). Then there exists some $Z^0 \in \mathcal{Z}$ such that

$$Eu_i(Z_i^0) < Eu_i(Y_i) - \varepsilon_i, \quad i = 1, \dots, n.$$

Consider that $y_i^* = Eu_i(Z_i^0)$, $i = 1, \dots, n$. Then $y_i^* < Eu_i(Y_i)$, $\lambda_i^{-1} = Eu_i(Y_i) - y_i^* = Eu_i(Y_i) - Eu_i(Z_i^0) > \varepsilon_i \geq 0$ and hence $\lambda_i \varepsilon_i < 1$ for $i = 1, \dots, n$. By the assumption that Y is a $\max_{i=1,n} \lambda_i \varepsilon_i$ -optimal solution of (2.2), $\forall Z \in \mathcal{Z}$

we have that

$$\begin{aligned} \max_{i=1,n} \lambda_i [Eu_i(Z_i) - Eu_i(Z_i^0)] &\geq \max_{i=1,n} \lambda_i [Eu_i(Y_i) - Eu_i(Z_i^0)] - \max_{i=1,n} \lambda_i \varepsilon_i = \\ &= 1 - \max_{i=1,n} \lambda_i \varepsilon_i > 0. \end{aligned}$$

But for $Z = Z^0$ we have in the last inequality $0 > 0$, false.

Lemma 1. *If $Y \in \mathcal{Z}$ is an ε -properly efficient portfolio of (2.1.) then the system*

$$\alpha_i Eu_i(Z_i) + \rho \sum_{j \in N} Eu_j(Z_j) < \alpha_i Eu_i(Y_i) + \rho \sum_{j \in N} Eu_j(Y_j) - \alpha_i \varepsilon_i - \rho \sum_{j \in N} \varepsilon_j, \quad i \in N \quad (2.3)$$

admits no solution $Z \in \mathcal{Z}$ for some $\rho > 0$, where $\alpha_i > 0, i \in N$.

Proof: Suppose that $Y \in \mathcal{Z}$ is an ε -properly efficient portfolio of (2.1). From the definition of this form of portfolio we have that the following system:

$$Eu_i(Z_i) \leq Eu_i(Y_i) - \varepsilon_i, \quad i = 1, \dots, n$$

admits no solution in \mathcal{Z} . In the same mode we have that

$$\alpha_i Eu_i(Z_i) \leq \alpha_i Eu_i(Y_i) - \alpha_i \varepsilon_i, \quad i = 1, \dots, n$$

admits no solution in \mathcal{Z} for any $\alpha_i > 0, i \in N$. Consider $\hat{Z} \in \mathcal{Z}$ be fixed.

We prove that if

$$\sum_{j \in N} Eu_j(Y_j) - \sum_{j \in N} \varepsilon_j \leq \sum_{j \in N} Eu_j(\hat{Z}_j)$$

then the system of the following inequalities is inconsistent for any $\rho > 0$,

$$\alpha_i Eu_i(\hat{Z}_i) + \rho \sum_{j \in N} Eu_j(\hat{Z}_j) < \alpha_i Eu_i(Y_i) + \rho \sum_{j \in N} Eu_j(Y_j) - \alpha_i \varepsilon_i - \rho \sum_{j \in N} \varepsilon_j, \quad i \in N.$$

This is because if it is not the case, we would have $\forall i \in N$

$$\alpha_i Eu_i(Y_i) - \alpha_i \varepsilon_i - \alpha_i Eu_i(\hat{Z}_i) > \rho \left[\sum_{j \in N} Eu_j(\hat{Z}_j) - \sum_{j \in N} Eu_j(Y_j) + \sum_{j \in N} \varepsilon_j \right] \geq 0$$

false, with the definition of the ε -properly efficient portfolio.

If

$$\sum_{j \in N} Eu_j(Y_j) - \sum_{j \in N} \varepsilon_j > \sum_{j \in N} Eu_j(\hat{Z}_j)$$

then define the following set:

$$I := \left\{ i \in N \mid Eu_i(Y_i) - \varepsilon_i > Eu_i(\hat{Z}_i) \right\} \neq \emptyset.$$

By first remark, Y is ε -efficient portfolio for (2.1). Then $\exists j_0 \in N$ such that $Eu_{j_0}(\hat{Z}_{j_0}) > Eu_{j_0}(Y_{j_0}) - \varepsilon_{j_0}$.

Let $Eu_l(\hat{Z}_l) - Eu_l(Y_l) + \varepsilon_l = \max_{i=1, n} \left[Eu_i(\hat{Z}_i) - Eu_i(Y_i) + \varepsilon_i \right] > 0$, so, by the definition of the ε -properly efficient portfolio exists $M > 0$, such that for any $j \in N$ satisfying $Eu_j(\hat{Z}_j) > Eu_j(Y_j) - \varepsilon_j$ and

$$\left[Eu_i(Y_i) - \varepsilon_i - Eu_i(\hat{Z}_i) \right] / \left[Eu_j(\hat{Z}_j) - Eu_j(Y_j) + \varepsilon_j \right] \leq M$$

or, equivalently,

$$\begin{aligned} \left[Eu_i(Y_i) - \varepsilon_i - Eu_i(\hat{Z}_i) \right] &\leq M \left[Eu_j(\hat{Z}_j) - Eu_j(Y_j) + \varepsilon_j \right] \\ &\leq M \left[Eu_l(\hat{Z}_l) - Eu_l(Y_l) + \varepsilon_l \right] \end{aligned}$$

and, hence,

$$\left[Eu_l(\hat{Z}_l) - Eu_l(Y_l) + \varepsilon_l \right]^{-1} \sum_{i \in I} \left[Eu_i(Y_i) - \varepsilon_i - Eu_i(\hat{Z}_i) \right] \leq M(m-1).$$

We have

$$0 < \sum_{i \in N} \left[Eu_i(Y_i) - \varepsilon_i - Eu_i(\hat{Z}_i) \right] \leq \sum_{i \in I} \left[Eu_i(Y_i) - \varepsilon_i - Eu_i(\hat{Z}_i) \right]$$

so, from the above inequality we obtain:

$$\left[Eu_l(\hat{Z}_l) - Eu_l(Y_l) + \varepsilon_l \right]^{-1} \sum_{i \in N} \left[Eu_i(Y_i) - \varepsilon_i - Eu_i(\hat{Z}_i) \right] \leq M(m-1).$$

Let be $\rho \leq \left(\min_{i \in N} \alpha_i \right) [M(m-1)]^{-1}$. Then

$$\begin{aligned} \rho &\leq \alpha_l [M(m-1)]^{-1} \\ &\leq \alpha_l \left[Eu_l(\hat{Z}_l) - Eu_l(Y_l) + \varepsilon_l \right] \left\{ \sum_{i \in N} \left[Eu_i(Y_i) - \varepsilon_i - Eu_i(\hat{Z}_i) \right] \right\}^{-1} \end{aligned}$$

or,

$$\rho \left\{ \sum_{i \in N} \left[Eu_i(Y_i) - \varepsilon_i - Eu_i(\hat{Z}_i) \right] \right\} \leq \alpha_l \left[Eu_l(\hat{Z}_l) - Eu_l(Y_l) + \varepsilon_l \right].$$

In this way we have the result:

$$\alpha_l Eu_l(\hat{Z}_l) + \rho \sum_{i \in N} Eu_i(Y_i) - \alpha_l \varepsilon_l - \rho \sum_{i \in N} \varepsilon_i \leq \alpha_l Eu_l(\hat{Z}_l) + \rho \sum_{i \in N} Eu_i(\hat{Z}_i)$$

therefore, the system of m inequalities

$$\alpha_i Eu_i(\hat{Z}_i) + \rho \sum_{j \in N} Eu_j(\hat{Z}_j) < \alpha_i Eu_i(Y_i) + \rho \sum_{j \in N} Eu_j(Y_j) - \alpha_i \varepsilon_i - \rho \sum_{i \in N} \varepsilon_i, \quad i \in N$$

admits no solutions. If we note that $\hat{Z} \in \mathcal{Z}$ can be any element of \mathcal{Z} we can conclude that the system

$$\alpha_i Eu_i(Z_i) + \rho \sum_{j \in N} Eu_j(Z_j) < \alpha_i Eu_i(Y_i) + \rho \sum_{j \in N} Eu_j(Y_j) - \alpha_i \varepsilon_i - \rho \sum_{i \in N} \varepsilon_i, \quad i \in N$$

has no solution for some $\rho > 0$.

If we put together the last two lemmas we can formulate the following theorem what gives the condition for a feasible allocation to be an ε -properly efficient portfolio of (2.1):

Theorem 2. *Any ε -properly efficient portfolio Y of (2.1) is an ε_0 -optimal portfolio to the optimization problem*

$$\min \varphi(Z), Z \in \mathcal{Z}, \quad (2.4)$$

where $\varphi(Z) = \max_{i \in N} \lambda_i \left\{ \left(Eu_i(Z_i) - y_i^* \right) + \rho \sum_j \left(Eu_j(Z_j) - y_j^* \right) \right\}$, for some $\rho >$

0 and $\varepsilon_0 = \max_{i \in N} \lambda_i \left(\varepsilon_i + \rho \sum_{i \in N} \varepsilon_i \right)$,

$\lambda_i = \left[\left(Eu_i(Y_i) - \rho \sum_{i \in N} \varepsilon_i \right) + \rho \sum_j \left(Eu_j(Y_j) - y_j^* \right) \right]^{-1}$ and y_i^* is a number such that $\lambda_i > 0, \forall i \in N$.

Proof: We consider $\alpha_i = 1, \forall i \in N$. By the above lemma there exists a $\rho > 0$ such that the system (2.3) admits no solution. By the first theorem, for any $\delta_1, \delta_2, \dots, \delta_n$ negative real numbers there exist positive numbers $\lambda_i = -\delta_i^{-1} > 0, i = \overline{1, n}$, such that $\forall Z \in \mathcal{Z}$

$$\max_{i=\overline{1, n}} \lambda_i \left[Eu_i(Z_i) - Eu_i(Y_i) + \rho \sum_j (Eu_j(Z_j) - Eu_j(Y_j)) + \varepsilon_i + \rho \sum_{i \in N} \varepsilon_i - \delta_i \right] \geq 1.$$

We can consider y_i^* such a number for that we have

$$\delta_i = y_i^* - Eu_i(Y_i) + \rho \sum_{j \in N} (y_j^* - Eu_j(Y_j)) < 0, i = \overline{1, n}.$$

If we denote $\lambda_i = -\delta_i^{-1} > 0, i = \overline{1, n}$, for any $Z \in \mathcal{Z}$ we have

$$\begin{aligned} 1 &\leq \max_{i=\overline{1, n}} \lambda_i \left[(Eu_i(Z_i) - y_i^*) + \rho \sum_{j \in N} (Eu_j(Z_j) - y_j^*) + \varepsilon_i + \rho \sum_{i \in N} \varepsilon_i \right] \\ &\leq \max_{i=\overline{1, n}} \lambda_i \left[(Eu_i(Z_i) - y_i^*) + \rho \sum_{j \in N} (Eu_j(Z_j) - y_j^*) \right] + \max_{i=\overline{1, n}} \lambda_i \left(\varepsilon_i + \rho \sum_{i \in N} \varepsilon_i \right). \end{aligned}$$

So, for any $Z \in \mathcal{Z}$ the proof is completed by

$$\begin{aligned} &\max_{i=\overline{1, n}} \lambda_i \left[(Eu_i(Z_i) - y_i^*) + \rho \sum_{j \in N} (Eu_j(Z_j) - y_j^*) \right] \\ &\geq \max_{i=\overline{1, n}} \lambda_i \left[(Eu_i(Y_i) - y_i^*) + \rho \sum_{j \in N} (Eu_j(Y_j) - y_j^*) \right] - \varepsilon_0. \end{aligned}$$

Remark 3. When $\varepsilon = 0$ the above theorem reduces to the theorem proved above about a properly efficient portfolio.

In the following we present another necessary condition for ε -properly efficient portfolio.

Theorem 3. If $Y \in \mathcal{Z}$ is an ε -properly efficient portfolio of (2.1) then there exists some $\rho > 0$ such that Y is ε_0 -optimal to the optimization problem

$$\min \varphi(Z), Z \in \mathcal{Z} \tag{2.5}$$

where $\varphi(Z) = \max_{i \in N} \lambda_i \left\{ (Eu_i(Z_i) - y_i^*) + \rho \sum_j (Eu_j(Z_j) - y_j^*) \right\}$, and $\varepsilon_0 =$

$\max_{i \in N} \lambda_i \left(\varepsilon_i + \rho \sum_{j \in N} \varepsilon_j \right)$, $\lambda_i = [Eu_i(Y_i) - y_i^*]^{-1}$ and y_i^* is a fixed number such that $\lambda_i > 0, \forall i \in N$.

Proof: Suppose that $Y \in \mathcal{Z}$ is an ε -properly efficient portfolio of (2.1) and let be y_i^* is a fixed number such that $\lambda_i = [Eu_i(Y_i) - y_i^*]^{-1} > 0, \forall i \in N$. By the above lemma there exists a $\rho > 0$ such that for any $Y \in \mathcal{Z}$ the following system is inconsistent:

$$\begin{aligned} & \lambda_i \left\{ (Eu_i(Z_i) - y_i^*) + \rho \sum_j (Eu_j(Z_j) - y_j^*) \right\} \\ < \lambda_i \left\{ (Eu_i(Y_i) - y_i^*) + \rho \sum_j (Eu_j(Y_j) - y_j^*) - \lambda_i \varepsilon_i - \rho \sum_{j \in N} \varepsilon_j \right\}, \forall i \in N. \end{aligned}$$

Hence, there exists some $i_0 \in N$ such that

$$\begin{aligned} & \lambda_{i_0} \left\{ (Eu_{i_0}(Z_{i_0}) - y_{i_0}^*) + \rho \sum_j (Eu_j(Z_j) - y_j^*) \right\} \\ & \geq \lambda_{i_0} \left\{ (Eu_{i_0}(Y_{i_0}) - y_{i_0}^*) + \rho \sum_j (Eu_j(Y_j) - y_j^*) - \lambda_{i_0} \varepsilon_{i_0} - \rho \sum_{j \in N} \varepsilon_j \right\} \\ & = 1 + \rho \sum_j (Eu_j(Y_j) - y_j^*) - \left(\lambda_{i_0} \varepsilon_{i_0} + \rho \sum_{j \in N} \varepsilon_j \right) \\ & = \max_{i=1,n} \lambda_i \left\{ (Eu_i(Y_i) - y_i^*) + \rho \sum_j (Eu_j(Y_j) - y_j^*) \right\} - \left(\lambda_{i_0} \varepsilon_{i_0} + \rho \sum_{j \in N} \varepsilon_j \right) \\ & = \max_{i=1,n} \lambda_i \left\{ (Eu_i(Y_i) - y_i^*) + \rho \sum_j (Eu_j(Y_j) - y_j^*) \right\} - \max_{i=1,n} \left[\lambda_i \varepsilon_i + \rho \sum_{j \in N} \varepsilon_j \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & \max_{i=1,n} \lambda_i \left\{ (Eu_i(Z_i) - y_i^*) + \rho \sum_j (Eu_j(Z_j) - y_j^*) \right\} \\ & \geq \max_{i=1,n} \lambda_i \left\{ (Eu_i(Y_i) - y_i^*) + \rho \sum_j (Eu_j(Y_j) - y_j^*) \right\} - \max_{i=1,n} \left[\lambda_i \varepsilon_i + \rho \sum_{j \in N} \varepsilon_j \right]. \end{aligned}$$

We can conclude that Y is an ε_0 -optimal portfolio of (2.5) and the proof is completed.

Remark 4. When $\varepsilon = 0$ the above theorem reduces to the theorem about properly efficient portfolio.

3. Sufficient conditions for ε -efficient portfolios

In the following section we present with two sufficient conditions for a feasible portfolio to be an ε -efficient portfolio of (2.1).

Theorem 4. *Let be $\varepsilon_0 = \max_{i \in N} \lambda_i \left(\varepsilon_i + \rho \sum_{j \in N} \varepsilon_j \right)$ and $Y \in \mathcal{Z}$ is an ε -efficient portfolio of (2.1) if there exists $\rho > 0$ such that for any y_i^* satisfying $\lambda_i := \left\{ (Eu_i(Y_i) - y_i^*) + \rho \sum_j (Eu_j(Y_j) - y_j^*) \right\}^{-1} > 0, \forall i \in N, Y \in \mathcal{Z}$ is an ε_0 -optimal portfolio to (2.4).*

Proof: Suppose that there exists $\rho > 0$ such that for any y_i^* satisfying $\lambda_i := \left\{ (Eu_i(Y_i) - y_i^*) + \rho \sum_j (Eu_j(Y_j) - y_j^*) \right\}^{-1} > 0, \forall i \in N, Y \in \mathcal{Z}$ is an ε_0 -optimal portfolio to (2.4). We assume that $Y \in \mathcal{Z}$ is not an ε -efficient portfolio of (2.1). By definition of an ε -efficient portfolio to (2.1) there exist $Z^0 \in \mathcal{Z}$ such that

$$Eu_i(Z_i^0) \leq Eu_i(Y_i) - \varepsilon_i, \quad i = 1, \dots, n$$

with at least one strict inequality.

We take $y_i^* = Eu_i(Z_i^0), i = 1, \dots, n$. Then

$$\begin{aligned} \lambda_i^{-1} &:= (Eu_i(Y_i) - y_i^*) + \rho \sum_j (Eu_j(Y_j) - y_j^*) \\ &= (Eu_i(Y_i) - Eu_i(Z_i^0)) + \rho \sum_j (Eu_j(Y_j) - Eu_j(Z_j^0)) \\ &> \varepsilon_i + \rho \sum_j \varepsilon_j, \quad i = 1, \dots, n. \text{ and we get} \end{aligned}$$

$$\lambda_i \left(\varepsilon_i + \rho \sum_j \varepsilon_j \right) < 1, \quad i = 1, \dots, n$$

hence

$$\varepsilon_0 := \max_{i \in N} \lambda_i \left(\varepsilon_i + \rho \sum_{j \in N} \varepsilon_j \right) < 1.$$

But $Y \in \mathcal{Z}$ is an ε_0 -optimal portfolio to (2.4), so, we have

$$\begin{aligned} 0 &= \max_{i \in N} \lambda_i \left[Eu_i(Z_i^0) - y_i^* + \rho \sum_j (Eu_j(Z_j^0) - y_j^*) \right] \\ &\geq \max_{i \in N} \lambda_i \left[Eu_i(Y_i) - y_i^* + \rho \sum_j (Eu_j(Y_j) - y_j^*) \right] - \varepsilon_0 \\ &= 1 - \varepsilon_0 > 0, \end{aligned}$$

which is false, so, $Y \in \mathcal{Z}$ is an ε -efficient portfolio of (2.1).

Theorem 5. *Let be $\varepsilon_0 = \max_{i \in N} \lambda_i \left(\varepsilon_i + \rho \sum_{j \in N} \varepsilon_j \right)$ and $Y \in \mathcal{Z}$ is an ε -efficient portfolio of (2.1) if there exists $\rho > 0$ such that for any y_i^* satisfying $\lambda_i := [Eu_i(Y_i) - y_i^*]^{-1} > 0, \forall i \in N, Y \in \mathcal{Z}$ there is an ε_0 -optimal portfolio to (2.5).*

Proof: Suppose that there exists $\rho > 0$ such that for any y_i^* satisfying $\lambda_i := [(Eu_i(Y_i) - y_i^*)]^{-1} > 0, \forall i \in N, Y \in \mathcal{Z}$ there is an ε_0 -optimal portfolio to (2.5). We assume that $Y \in \mathcal{Z}$ is not an ε -efficient portfolio of (2.1). By definition of an ε -efficient portfolio to (2.1) there exist $Z^0 \in \mathcal{Z}$ such that

$$Eu_i(Z_i^0) \leq Eu_i(Y_i) - \varepsilon_i, \quad i = 1, \dots, n \quad (3.1)$$

with at least one strict inequality. We take $y_i^* = Eu_i(Z_i^0) - \varepsilon_i, i = 1, \dots, n$.

Then $\lambda_i^{-1} := (Eu_i(Y_i) - y_i^*) = \varepsilon_i > 0, i = 1, \dots, n$ and hence

$$\varepsilon_0 := \max_{i \in N} \lambda_i \left(\varepsilon_i + \rho \sum_{i \in N} \varepsilon_i \right) = 1 + \rho \sum_{i \in N} \varepsilon_i.$$

By relation (3.1) and the ε_0 -optimality of Y with respect to (2.5) we have:

$$\begin{aligned} 0 &> \max_{i \in N} \lambda_i \left[Eu_i(Z_i^0) - y_i^* + \rho \sum_j (Eu_j(Z_j^0) - y_j^*) \right] \\ &> \max_{i \in N} \lambda_i \left[Eu_i(Y_i) - y_i^* + \rho \sum_j (Eu_j(Y_j) - y_j^*) \right] - \varepsilon_0 \\ &= \max_{i \in N} \lambda_i \left(\varepsilon_i + \rho \sum_{i \in N} \varepsilon_i \right) - \varepsilon_0 \\ &= \max_{i \in N} \left(1 + \rho \sum_{i \in N} \varepsilon_i \right) - \varepsilon_0 = 0, \end{aligned}$$

which is false. Therefore, Y is an ε -efficient portfolio to (2.1).

4. A necessary and sufficient condition for ε -weak efficient portfolios

In the following theorem we characterize an ε -weak minimum portfolio for the problem (2.1).

Definition 5. *$Y \in \mathcal{Z}$ is said to be a Pareto minimum portfolio or simply a minimum portfolio of (2.1) if*

$$Eu_i(Y_i) \not\leq Eu_i(Z_i), \text{ for any } Z \in \mathcal{Z}, Eu_i(Z_i) \neq 0, \forall i = \overline{1, n}.$$

Definition 6. $Y \in \mathcal{Z}$ is said to be a ε -weak minimum portfolio of (2.1) if

$$Eu_i(Y_i) \not\geq Eu_i(Z_i) + \varepsilon_i, \text{ for any } Z \in \mathcal{Z}.$$

Theorem 6. $Z^0 \in \mathcal{Z}$ is an ε -weak minimum portfolio for the problem (2.1) if and only if exists $\mu \in \mathbb{R}_+^n$ with $\sum_{i=1}^n \mu_i = 1$ such that Z^0 is an ε minimum portfolio for the following minimization problem

$$\min \sum_{i \in N} \mu_i Eu_i(Z_i) \text{ subject to } Z \in \mathcal{Z}. \quad (4.1)$$

Proof: If $Z^0 \in \mathcal{Z}$ is an ε -weak minimum portfolio for the problem (2.1) then the following system

$$Eu_i(Z_i) - Eu_i(Z_i^0) + \varepsilon_i \leq 0$$

admits no solution. By standard separation argument there exists $\mu \in \mathbb{R}_+^n$, nonzero, such that

$$\sum_{i \in N} \mu_i [Eu_i(Z_i) - Eu_i(Z_i^0) + \varepsilon_i] \geq 0, \quad Z \in \mathcal{Z}.$$

As $\mu \neq 0$ we can consider μ such that $\sum_{i=1}^n \mu_i = 1$ and $\sum_{i=1}^n \mu_i \left(\sum_{j=1}^n \varepsilon_j \right) = \varepsilon$ holds. So, we can note that

$$\sum_{i \in N} \mu_i Eu_i(Z_i) - \sum_{i \in N} \mu_i Eu_i(Z_i^0) \geq -\varepsilon.$$

Hence Z^0 is an ε minimum portfolio for the minimization problem (4.1).

Conversely, suppose that there exists $\mu \in \mathbb{R}_+^n$ with $\sum_{i=1}^n \mu_i = 1$ such that Z^0 is an ε minimum portfolio for the minimization problem (4.1). We claim that Z^0 is an ε -weak minimum portfolio for (2.1). If this does not happen then there exists a portfolio $Y \in \mathcal{Z}$ such that

$$Eu_i(Y_i) - Eu_i(Z_i^0) + \varepsilon_i < 0.$$

Since $\mu \in \mathbb{R}_+^n$ with $\sum_{i=1}^n \mu_i = 1$

$$\sum_{i \in N} \mu_i Eu_i(Y_i) - \sum_{i \in N} \mu_i Eu_i(Z_i^0) < -\varepsilon$$

holds, which come in contradiction with the fact that Z^0 is an ε minimum portfolio for (4.1).

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