VARIATION OF CONSTANT SUM CONSTRAINT FOR INTEGER MODEL WITH NON UNIFORM VARIABLES

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Abstract

This paper wants to continue the studies for a class of problems named constant sum integer programming introduced earlier. This approach tries to see what is happening when a variation appears in right hand of constant sum condition. In this paper I prove that limited variation influence no more than 3 model variables in model optimum and there is a way to say which these variables are. My consideration could be important if we want to transform a constant sum condition in a stochastic form.

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1. **Introduction**

In earlier paper I had considered constant sum integer model with non uniform variables which had explicit form:

(MSCDN)
$$\begin{cases} \text{opt } cx \\ Ax \le b \\ 1 \bullet x = P \\ x_i \in \{0, 1, ..., p_i\}, \mathrm{i} = 1, ..., \mathrm{n} \end{cases}$$

In such a model $1 \bullet X = P$ (or $x_1 + x_2 + ... + x_n = P$) is constant sum constraints.

If $c=(c_1,c_2,...,c_n)$ has the property of monotony ($c_1 \le c_2 \le ... \le c_n$) and there exists x $\in$ R^n, for which $Ax \le b$ and $1 \bullet X = P$ (x $\in$ R^n is a feasible solution for (MSCDN)), then in a lexicographic suitable order the smallest feasible solution x^* is optimal solution for (MSCDN).

In the same paper I gave an algorithm to generate optimal solution for (MSCDN)

In the above condition, the algorithm for (MSCDN) produced an optimal solution of the form

$$x^* = (p_1, p_2, ..., p_k, \alpha, 0, ..., 0), \tag{1}$$

where for the value in position $k+1$ I have $0 \le \alpha < p_{k+1}$. Also it was proved that this result is a generalization for the case of 0-1 models with monotone coefficients in goal function.

2. **A family of constant sum integer model**

I wish to continue the study about constant sum integer model. I consider a family of constant sum integer programs in which I have a variation for the value P of constant sun constraints. To simplify the model I consider that monotony condition are fulfill for coefficients. So, all the models will have an optimal solution of form (1).

The variation considered so far must be in maximum length of p_l. It is also assumed that in this variation the constant sum integer model still have optimal solutions.

Starting with (MSCDN) model I can build the family of models (MSCDN$_{q_j}$):

(MSCDN$_{q_j}$)
$$\begin{cases} \text{opt}\, cx \\ Ax \le b \\ 1 \bullet x = q_j \\ x_i \in \{0, 1, ..., p_i\}, \mathrm{i} = 1, ..., \mathrm{n} \end{cases}$$

where

$$q_j \in [P-t, P+t] \cap N,$$

and

$$t \le \min_{i=\overline{1,n}} p_i$$

is chosen so that for any, $y \in [P-t, P+t] \cap N$ the (MSCDN$_{q_j}$) model has an optimal solution.

3. **Optimal solution variation for the family of constant sum integer model**

The mail goal of this paper is to set a result about optimal solution variation for the family presented in paragraph II.

First I must observe that all members of (MSCDN$_{q_j}$) model family has an optimal solution for which there is a permutation σ so that $\left(c_{\sigma(i)}\right)$ is a monotone string and $\left(x^*_{\sigma(i)}\right)$ is of form (1).

This observation allows me to give the next result:

Theorem. Let (MSCDN$_{q_j}$) be a family of models as it is defined in paragraph II. Then there exists $u, v, w \in \{1, 2, ..., n\}$ so that with exception of positions u, v and w, optimal solutions are invariant. In addition, if σ is a permutation so that $\left(c_{\sigma(i)}\right)$ are ordered, then $\sigma(u)$, $\sigma(v)$ and $\sigma(w)$ are three consecutive integers.

Proof. The general case can be reduced to the one of ordered string for the value of $c_1, c_2, ..., c_n$ Let σ be the permutation which ordered initial values of coefficient in objective function.

Let x^*_p be the optimal value obtained for value P as a right hand value in constant sum constraint. Now, using known properties for model family (MSCDN$_{q_j}$), we have

$$x^*_p = (p_1, p_2, ..., p_k, \alpha, 0, ..., 0)$$

and this solution is given by algorithm specified in [2].

For an arbitrary $\overline{q} \in [-t, t] \cap N$ I have 3 cases.

Case I. For $\alpha + \overline{q} \geq p_{k+1}$

I use again the algorithm given in [2] to reach a solution for P as a right hand of constant sum constraint, which exists because the family (MSCDN$_{q_j}$) has this property. Then, there exists an integer k so that

$$p_1 + p_2 + ... + p_k \leq P$$

and

$$p_1 + p_2 + ... + p_k + p_{k+1} > P$$

and

$$\alpha = P - p_1 - p_2 - ... - p_k$$

By adding $\overline{q}$ to the first two relations written above I obtain

$$p_1 + p_2 + ... + p_k + \overline{q} \le P + \overline{q}$$

and

$$p_1 + p_2 + ... + p_k + p_{k+1} + \overline{q} > P + \overline{q} .$$

In our case condition, $\alpha + \overline{q} \ge p_{k+1}$ it follows that

$$p_1 + p_2 + ... + p_k + p_{k+1} \le P + \overline{q}$$

and

$$p_1 + p_2 + ... + p_k + p_{k+1} + p_{k+2} > P + \overline{q} .$$

If I consider $\beta = p_{k+1} - \alpha - \overline{q}$, than the solution for $P + \overline{q}$ as a right hand of constant sum constraint is

$$x^*_{p+\overline{q}} = \left(p_1, p_2, ..., p_k, p_{k+2}, \beta, 0, ..., 0\right)$$

and so, new values appear only in positions $k+1$ and $k+1$.

Case II. For $\alpha + \overline{q} < 0$

By similar consideration, new solution is

$$x^*_{p+\overline{q}} = \left(p_1, p_2, ..., p_{k-1}, \beta, 0, ..., 0\right),$$

so, new values appear in positions k and $k+1$.

Case III. For $0 \le \alpha + \overline{q} < p_{k+1}$

This is a similar case to case I and only value or rang $k+1$ is modified.

The conclusion for these three cases, for any values $\overline{q} \in [-t, t] \cap N$, no more than three values are modified in solution vector, the values of rang k, $k+1$ and $k+2$. I must remember that objective function coefficients are ordered.

Coming back to general situation, optimal solution is reached by applying σ^{-1} to optimal solution in ordered coefficient case. So we have only 3 non invariant component in general solution x^*, placed in position u, v and w which is $\sigma^{-`}(k)$, $\sigma^{-1}(k+1)$ and $\sigma^{-1}(k+2)$. And so I finish the theorem demonstration.

4. **Optimal solution variation subspace**

Now, because I know that variation for right hand of constant sum constraint produce modification for no more than three values in optimal solution, it is possible to make a projection of solution space

$$S_0 \subset S_f = \prod_{i=1}^{n} \{0,1,...,p_i\}$$

into 3-dimension space

$$\overline{S}_f = \{0,1,...,p_u\} \times \{0,1,...,p_v\} \times \{0,1,...,p_w\},$$

where u, v and w are modified value rang in optimal solution so that u<v<w. If projection space is $\overline{S}_0 \subset \overline{S}_f$, the goal for this section is to determine projection space form.

If I follow the demonstration for the above theorem, for ordered coefficients model I observe that if q_j modification produces variation only in x^*_{k+1} then the value for this element is between 0 and $p_{\sigma^{-1}(k+1)}$. So I have that

$$(p_u, y, 0) \in \overline{S}_0,\ p_u = p_{\sigma^{-1}(k)},\ y \in \{0,1,...,p_v\} \text{ with } p_v = p_{\sigma^{-1}(k+1)}$$

and

$$P + \overline{q} \in [-\alpha, p_v - \alpha] \cap [P - t, P + t] \cap N$$

represent right hand in constant sum constraint.

In a similar way, if modified values in ordered coefficients are of rang k and k+1 then

$$(z,0,0) \in \overline{S}_0,\ z \in \{p_u - t + \alpha, ..., p_u\}.$$

Also, if rang k+1 and k+2 values are modified, then

$$(p_u, p_v, s) \in \overline{S}_0,\ s \in \{0,1,...,P + t - p_v\}.$$

Now I can give the following result.

Lemma. If (MSCDN $_{q_j}$) is a family of models defined in section II with optimal solution space

$$S_0 \subset S_f = \prod_{i=1}^{n} \{0,1,...,p_i\}$$

and non invariant optimal solution values of rang u, v, w so that u<v<w then

$$\mathrm{pr}_{\overline{S}_f}\ \overline{S}_0 = \{(x,0,0) | x \in \{p_u - t + \alpha, ..., p_u\}\} \cup \{(p_u, y, 0) | y \in \{0,1,...,p_v\}\} \cup$$
$$\cup \{(p_u, p_v, z) | z \in \{0,1,...,P + t - p_v\}\}$$

5. **Conclusions**

In the real world, it is the best model formulation for a real problem in the field of stochastic model.

Till now, all studies about percentage programming are in respect with deterministic models.

This paper shows that a significant part of optimal solution for constant sum integer model is invariant under the above assumption. I consider that this result is a preliminary one and it prepares the future studies in which to have a constant sum stochastic integer programming.

Such a model can have a value for constant sum which is a discrete random variable with some specific probabilistic distribution. Another stochastic model in this area can have an estimated value for constant sum. Both subjects will be considered later.

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