SOME GENERALIZATIONS OF F-CONNECTIONS ON DIFFERENTIABLE MANIFOLDS

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Abstract
In this article we generalize the notion of F-connection, where F is an almost product structure on a manifold and we give some characterizations of the connections compatible with those structures.

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1. **Introduction**

Differentiable manifolds with almost product structures (ap)-structures, together with other important structures were investigated by G. Walker [1], Wilmore [2], and Yano [6]. New properties of these structures were established by Legrand [3] and Hsu [4].

In this paper we generalize the notion of *F*-connection for a connection ∇ on a manifold, where *F* is an (ap)-structure, by generalizing the (ap)-structures. We also establish some properties of these connections.

Let *M* be a differentiable manifold of dimension *n* and T_pM the tangent space to *M* in the point *p* with $p \in M$. We denote with Γ(*TM*) the tangent bundle of *M*, with F(*M*) the ring of real functions on *M*, with $D_s^r(M)$ the F(*M*) module of (*r, s*)-tensor fields, with C(*M*) the set of connections on *M,* and with $D^1(M)$ the F(*M*) module of sections of *M*.

Let *F* be an almost product (ap)-structure on *M*, that means *F* is an (1, 1)-tensor field defined on *M* satisfying $F^2 = I$, where I is the identity. Setting:

$$\Psi_F(\nabla)_X = \frac{1}{2}\left(\nabla_X + F \circ \nabla_X \circ F\right) \text{ and} \quad (1)$$

$$\chi_F(\tau)=\frac{1}{2}(\tau_X+F\circ\tau_X\circ F),$$

where τ is an (1, 2)-tensor field on *M*, $\forall X\in D^1(M)$ it follows

$$\Psi_F(\nabla)\in C(M),\ \chi_F(\tau)\in D_2^1(M) \tag{2}$$

and

$$\Psi_F^2=\Psi_F,\chi_F^2=\chi_F,\ \Psi_F(\nabla+\tau)=\Psi_F(\nabla)+\chi_F(\tau). \tag{3}$$

Hence ψ_F is the F(*M*)-affine projector on C(*M*), associated to the F(*M*)-linear projector χ_F on $D_2^1(M)$.

Definition 1. *A connection* ∇ *on M is called compatible with the (ap)-structure F (or is an F-connection) if it satisfies* $\nabla F=0$*, where*

$$(\nabla_X F)(Y)=\nabla_X F(Y)-F(\nabla_X Y)$$

for all X, Y∈ Γ(TM).

We will denote by $C_F(M)$ the set of all connections compatible with F. From the expression of $\Psi_F(\nabla)$ it follows that $\Psi_F(\nabla)_X(F)=0$, $\forall X\in D^1(M)$, i.e. the image of any connection ∇ by the projector Ψ_F is an F-connection.

Conversely, if $\nabla_X F=0$ it follows that $\nabla_X\circ F-F\circ\nabla_X=0$ and so $\Psi_F(\nabla)_X=\nabla_X,\forall X\in D^1(M)$ i.e. $\nabla\in \operatorname{Im}\Psi_F$.

Thus we have:

Theorem 1. *The set* $C_F(M)$ *of the connections compatible with the (ap)-structure F is the affine submodule of C(M) which is the image of the affine projector* Ψ_F*, i.e.* $\operatorname{Im}\Psi_F=C_F(M)$

Let ∇^0 be a fixed connection on M. Since $\operatorname{Im}\Psi_F=C_F(M)$, then for each connection $\nabla\in C_F(M)$ there exists $\nabla'\in C(M)$ such that $\nabla'=\Psi_F(\nabla)$. But from [5], there exists $\tau\in D_2^1(M)$ such that $\nabla'=\nabla^0+\tau$ and therefore, $\nabla=\Psi_F(\nabla^0+\tau)$.

Then it follows:

Theorem 2. *The set* $C_F(M)$ *of the connections* ∇ *compatible with the (ap)-structure F is given by*

$$\nabla = \Psi_F(\nabla^0) + \chi_F(\tau). \tag{4}$$

where ∇^0 *is a fixed connection on M and τ is an arbitrary (1, 2)-tensor field on M.*

2. **Main results**

Let us consider F an (1, 1)-tensor field on M, such that $F^3 = F$ and $F^2 \neq I$. We will try to construct Ψ_F such that Ψ_F is a connection on *M* and a similar result to the *Theorem 1* holds.

For that let $G = I - 2F^2$. We have that G is also an (1, 1)-tensor field on M and

$$G^2 = \left(I - 2F^2\right)^2 = I - 4F^2 + 4F^4 = I. \tag{5}$$

So *G* is an (ap)-structure on M.

Hence, if we set $\Psi'_F(\nabla)_X = \frac{1}{2}(\nabla_X + G \circ \nabla_X \circ G)$ we obtain that $\Psi'_F(\nabla)_X \in C(M)$.

Definition 2. *A connection* ∇ *on M is called compatible with the (1, 1)-tensor field* F^2 *(or is* F^2*-an connection) if it satisfies* $\nabla F^2 = 0$.

Theorem 3. *The set* $C_{F^2}(M)$ *of the connections compatible with the (1, 1)-tensor field* F^2 *is the affine submodule of C(M) which is the image of the affine projector* $\Psi'_F(\nabla)_X$, *i.e.* $\operatorname{Im}\Psi'_F(\nabla) = C_{F^2}(M)$.

Proof. First of all, we will prove that Ψ'_F takes any connection on *M* into an F^2-connection. We observe that $G \circ F^2 = F^2 \circ G = -F^2$.

Let ∇ be a connection on *M*. We will compute

$$\Psi'_F(\nabla)_X\left(F^2\right) = \Psi'_F \circ F^2 - F^2 \circ \Psi'_F.$$

We have, consecutively,

$$\Psi'_F \circ F^2 = \Psi'_F(\nabla)_X F^2(Y) = \frac{1}{2}\left[\nabla_X F^2(Y) + G\left(\nabla_X G\left(F^2(Y)\right)\right)\right] =$$

$$= \frac{1}{2}\left[\nabla_X F^2(Y) - G\left(\nabla_X F^2\right)\right] = \frac{1}{2}\left[\nabla_X F^2(Y) - \left(I - 2F^2\right)\nabla_X F^2(Y)\right] =$$

$$= \frac{1}{2}\left[\nabla_X F^2(Y) - \nabla_X F^2(Y) + 2F^2\left(\nabla_X F^2(Y)\right)\right] = F^2\left(\nabla_X F^2(Y)\right),$$

and

$$F^2 \circ \Psi'_F = F^2\left(\Psi'_F(\nabla)_X Y\right) = \frac{1}{2}\left[F^2(\nabla_X Y) + F^2\left(G\left(\nabla_X(Y)\right)\right)\right] =$$

$$= \frac{1}{2}\left[F^2(\nabla_X Y) - F^2\left(\nabla_X G(Y)\right)\right] = \frac{1}{2}\left[F^2(\nabla_X Y) - F^2\left(\nabla_X\left(Y - 2F^2(Y)\right)\right)\right] =$$

$$= \frac{1}{2}\left[F^2(\nabla_X Y) - F^2(\nabla_X Y) + F^2\left(\nabla_X F^2(Y)\right)\right] = F^2\left(\nabla_X F^2(Y)\right).$$

Therefore, $\Psi'_F(\nabla)_X\left(F^2\right) = 0$ and then the image of any connection ∇ by the projector Ψ'_F is an F^2-connection.

Now, let ∇ be an F^2-connection. Then

$$\Psi'_F(\nabla)_X = \frac{1}{2}\left(\nabla_X + G \circ \nabla_X \circ G\right) = \frac{1}{2}\left[\nabla_X + \left(I - 2F^2\right) \circ \nabla_X \circ \left(I - 2F^2\right)\right] =$$

$$= \frac{1}{2}\left[\nabla_X + \nabla_X - 2\nabla_X \circ F^2 - 2F^2 \circ \nabla_X + 4F^2 \circ \nabla_X \circ F^2\right] =$$

$$= \frac{1}{2}\left[2\nabla_X - 4F^2 \circ \nabla_X + 4F^4 \circ \nabla_X\right] = \nabla_X .$$

Hence $C_{F^2}(M) = \operatorname{Im}\Psi'_F$.

Let us consider F an (1, 1)-tensor field on *M* such that $F^{2n+1} = F$ and $F^{2n} \neq I$, where *n* is at least 2. We will try to generalize the construction of Ψ'_F such that Ψ'_F is a connection on M and a similar result to the *Theorem 3* holds.

For that, let $H = I - 2F^{2n}$. We have that *H* is also an (1, 1)-tensor field on *M* and

$$H^2 = \left(I - 2F^{2n}\right)^2 = I - 4F^{2n} + 4F^{4n} = I \; . \tag{6}$$

So H is an (ap)-structure on M.

Thus, if we set $\Psi''_F(\nabla)_X = \frac{1}{2}(\nabla_X + H \circ \nabla_X \circ H)$ we obtain that $\Psi''_F(\nabla)_X \in C(M)$.

Definition 3. *A connection* ∇ *on M is called compatible with the (1 ,1) tensor field* F^{2n} *(or is an* F^{2n} *-connection) if it satisfies* $\nabla F^{2n} = 0$ *.*

Theorem 4. *The set* $C_{F^{2n}}(M)$ *of the connections compatible with the (1,1)-tensor field* F^{2n} *is the affine submodule of C(M) which is the image of the affine projector* $\Psi''_F(\nabla)_X$ *, i.e.* $\Psi''_F(\nabla)_X = C_{F^{2n}}(M)$.

Proof. First of all, we will prove that Ψ''_F takes any connection on M into an F^{2n} - connection. We observe that: $H \circ F^{2n} = F^{2n} \circ H = -F^{2n}$.

Let ∇ be a connection on M. We will compute
$\Psi''_F(\nabla)_X\left(F^{2n}\right) = \Psi''_F \circ F^{2n} - F^{2n} \circ \Psi''_F$.

We have

$$\Psi''_F \circ F^{2n} = \Psi''_F(\nabla)_X F^{2n}(Y) = \frac{1}{2}\left[F^{2n}(\nabla_X Y) + H\left(\nabla_X H\left(F^{2n}(Y)\right)\right)\right] =$$
$$= \frac{1}{2}\left[F^{2n}(\nabla_X Y) - H\left(\nabla_X F^{2n}\right)\right] = \frac{1}{2}\left[F^{2n}(\nabla_X Y) - \left(I - 2F^{2n}\right)\nabla_X F^{2n}(Y)\right] =$$
$$= \frac{1}{2}\left[F^{2n}(\nabla_X Y) - \nabla_X F^{2n}(Y) + 2F^{2n}\left(\nabla_X F^{2n}(Y)\right)\right] = F^{2n}\left(\nabla_X F^{2n}(Y)\right).$$

and

$$F^{2n} \circ \Psi''_F = F^{2n}\left(\Psi''_F(\nabla)_X Y\right) = \frac{1}{2}\left[F^{2n}(\nabla_X Y) + F^{2n}\left(H\left(\nabla_X(Y)\right)\right)\right] =$$

$$=\frac{1}{2}\left[F^{2n}(\nabla_X Y)-F^{2n}(\nabla_X Y)+F^{2n}\left(\nabla_X F^{2n}(Y)\right)\right]=F^{2n}\left(\nabla_X F^{2n}(Y)\right).$$

So $\Psi''_F(\nabla)_X\left(F^{2n}\right)=0$ and then the image of any connection ∇ by the projector Ψ''_F is an F^{2n}-connection.

Now, let be ∇ an F^{2n}-connection. Then

$$\Psi''_F(\nabla)_X=\frac{1}{2}(\nabla_X+H\circ\nabla_X\circ H)=\frac{1}{2}\left[\nabla_X-\left(I-2F^{2n}\right)\circ\nabla_X\circ\left(I-2F^{2n}\right)\right]=$$

$$=\frac{1}{2}\left[\nabla_X+\nabla_X-2\nabla_X\circ F^{2n}-2F^{2n}\circ\nabla_X+4F^{2n}\circ\nabla_X\circ F^{2n}\right]=$$

$$=\frac{1}{2}\left[2\nabla_X-4F^{2n}\circ\nabla_X+4F^{4n}\circ\nabla_X\right]=\nabla_X.$$

Thus $C_{F^{2n}}(M)=\operatorname{Im}\Psi''_F$.

Remark 1. Once we have defined an almost product structure, F, on a differentiable manifold, *M*, then *M* carries two globally complementary distributions D and D'. We denote with V and V' respectively the projection tensor fields corresponding to the two distributions D and D'. We say that X is a vector field which belongs to the distribution D if for every point *p* in *M*, we have $X_p\in D_p(M)$, where $D_p(M)=V\left(T_pM\right)$.

Let ∇ be a linear connection on the manifold *M*. We say that the distribution D is parallel with respect to the connection ∇ if for every vector field X which belongs to the distribution D, the vector field $\nabla_Y X$ belongs to the distribution D for every vector field Y of the manifold *M*.

We can define on the manifold *M* two linear connections by:

$$Q_X Y=V\left(\nabla_X V(Y)\right)+V'\left(\nabla_X V'(Y)\right), \tag{7}$$

$$L_X Y=V\left(\nabla_{V(X)}V(Y)\right)+V'\left(\nabla_{V'(X)}V'(Y)\right)+V\left[V'X,VY\right]+V'\left[VX,V'Y\right]. \tag{8}$$

Proposition 1. *The distributions of the almost product structure F are both parallel with respect to the connections Q and L, for every connection* ∇ *.*

Proof. Let Y be a vector field which belongs to the distribution D of the almost product structure F. Then we have $V'(Y)=0$ and hence $V'(Q_X Y)=0, V'(L_X Y)=0$, for every vector field X on the manifold M.

Proposition 2. *The connection Q is equal to the connection* ∇ *if and only if the distributions of the almost product structure F are parallel with respect to the connection* ∇ *.*

Proof. If the connections Q and ∇ are equal then we get that

$$V\left(\nabla_X V'(Y)\right)+V'\left(\nabla_X V(Y)\right)=0. \tag{9}$$

By the properties of the projectors it follows that $V\left(\nabla_X V'(Y)\right)=0$ and $V'\left(\nabla_X V(Y)\right)=0$.

Hence, the distributions of F are parallel with respect to the connection ∇. The converse can be verified immediately.

Theorem 5. *If one of the connections Q or L is symmetric, then the almost product structure F is integrable.*

Proof. The almost product structure F is integrable if the distributions *D* and D' are involutive, that is if $V'(VX,VY)=0$ and $V(V'X,V'Y)=0$ for any two vector fields X, Y on the manifold *M*. If the connection Q is symmetric then we have that $Q_X Y - Q_Y X = [X,Y]$.

Thus the relation $V'(VX,VY)=0$ is equivalent to the condition

$$V'Q_{YX}VY - V'Q_{VY}VX = 0. \tag{10}$$

This condition is verified by the *Proposition 1*.

Analog if L is symmetric, using the relation $V(V'X,V'Y)=0$, the result is proved

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