THE CLARKE'S SUBDIFFERENTIAL FOR VECTOR VALUED FUNCTIONS

ROZOVEANU, Petru
Department of Mathematics, Bucharest,
Academy of Economic Studies,
rozespierre@yahoo.com

Abstract
In this paper we examine the construction of Clarke's derivative for vector-valued functions. We use another kind of Lipschitz functions, which allow us to leave the context of normed spaces. As base of the generalization we use the approach used by Clarke for real functions. For vector-valued functions, Clarke uses the Rademacher theorem, which is not available in general contexts.

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1. **Introductory notations and definitions**

Let F be an real ordered vector space, whose order relation is denoted by $\leqq$. If $a,\ b \in F,\ a \leqq b$, then we denote by $\left[a,\ b\right]$ the order segment $\left\{x \in F : a \leqq x \leqq b\right\}$. A set $A \subseteq F$ is called *full,* if from $a,\ b \in A$, $a \leqq b$ it follows $\left[a,b\right] \subseteq A$. If F is also a topological vector space and there exists a base of full neighborhoods of 0, then F is called a *topological ordered vector space*. The topological convergence is called τ*-convergence.*

An ordered vector space is called *archimedian* if for every $x \geqq 0$ we have:

$$\inf_n \frac{1}{n} x = 0 .$$

A normed lattice is an ordered vector space which is a lattice, and the norm satisfy the condition:

$$|x| \leqq |y| \Rightarrow \|x\| \leq \|y\| .$$

Every normed lattice is archimedian and has closed positive cone.

In an ordered vector space, the order convergence of a generalized sequence $(x_\delta)_{\delta\in\Delta}$ is called ω-convergence and it is defined as follows.

The sequence $(x_\delta)_{\delta\in\Delta}$ is ω-convergent to x if there exists two generalized sequence $(a_t)_{t\in T}$ and $(b_s)_{s\in S}$ such that the sequence a_t is increasing and has least upper bound x, the sequence b_s is decreasing and has greatest lower bound x, and for every $(t,s)\in T\times S$ there is $\delta_0\in\Delta$ such that for every $\delta\geq\delta_0$ we have:

$a_t \leqq x_\delta \leqq b_s$.

If F is a topological ordered vector space, we say that the topology of F is ω-continuous if every generalized sequence which is decreasing to 0 (hence ω-convergent to 0) is also τ-convergent to 0.

2. **A class of continuous functions**

We denote by E a topological vector space, and by F a topological ordered vector space. We denote also by F_+ the cone of positive elements of F.

Let $E_0 \subseteq E$ be any set, and $f: E_0 \to F$ be a vector valued function.

Using the topological structures of E and F, we obtain the usual notion of continuous function. But exists continuous functions which are not order bounded.

Example 1. *Let* $E_0=[0,1]$, $F=C[0,1]$ *with the usual order, and the norm* $\|x(\cdot)\|=\int_0^1 |x(t)|dt$. *Let* $f:[0,1]\to C[0,1]$ *be defined by* $f(0)=0$ *and for any* $x\in(0,1]$, *let* $f(x)$ *be a continuous function* $y(\cdot)$ *defined on the interval* $[0,1]$ *taking real values, and being such that:*

- $y\left(\left|\sin\frac{1}{x}\right|\right)=\frac{1}{x}$;
- $\int_0^1 |y(t)|dt = x$.

Then, for every $\varepsilon>0$, $f([0,\varepsilon])$ is not order bounded, but $\|f(x)\|=x\leq\varepsilon$, for every $x\in[0,\varepsilon]$, hence f is continuous in $x=0$, despite the fact that it is not order bounded in none of the neighborhoods of $x=0$.

On the other hand, we can take only the order structure of F and define the $\tau\omega$-continuous functions. For such a function, if $(x_\delta)_{\delta\in\Delta}$ is a generalized sequence which is τ-convergent to x^0 then $(f(x_\delta))_{\delta\in\Delta}$ is a generalized sequence which is ω-convergent to 0. But such functions are not always topologically bounded.

Example 2. *Let $E_0 = [0,1]$, and F be the space of those real functions defined on $[0,1]$ which have bounded derivative. We consider on F the usual order, and the norm defined by:*

$$\|x(\cdot)\| = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)|.$$

Let $f:[0,1]\to F$ be defined by $f(x)=0$, if $x \neq \frac{1}{n}$, and for $x=\frac{1}{n}$ let $f(x)\in F$ be a function defined on the interval $[0,1]$, in the following way:

$$f(x)(t) = \begin{cases} (nt-1)^2, \text{ for t} \in \left[0, \frac{1}{\text{n}}\right] \\ 0, \qquad\qquad \text{for t} \in \left(\frac{1}{\text{n}}, 1\right]. \end{cases}$$

Then $\left(f\left(\frac{1}{n}\right)\right)_n$ is a decreasing sequence to 0, and $\lim\limits_{n\to\infty}\left\|f\left(\frac{1}{n}\right)\right\| = +\infty$. Because for $x\neq\frac{1}{n}$ we have $f(x)=0$, it follows that f is ω-continuous in $x=0$, but it is not τ-bounded in none of the neighborhoods of $x=0$.

We suppose now that F is order complete. Combining in F the topological structure with the order structure, we introduce the following notion of continuous function.

For a neighborhood V of x^0 such that $h(V)$ is order bounded, we denote by $\sup h(V)$ the least upper bound of the set $h(V)$ and by $\inf h(V)$ the greatest lower bound of the same set. We obtain in this way two generalized sequences $(\sup h(V))_{V\in\mathcal{V}}$ and $(\inf h(V))_{V\in\mathcal{V}}$ of elements of F.

Definition 1. *A function $h: E_0 \to F$ is called m-continuous in $x^0 \in E_0$ if it is order bounded on a neighborhood of x^0 and:*

$$\tau-\lim_{V\in\mathcal{V}}\left(\sup h\left(V\right)\right)=\tau-\lim_{V\in\mathcal{V}}\left(\inf h\left(V\right)\right)=h\left(x^{0}\right).$$

The definition implicitly suppose the existence of limits involved.

We recall that if E is a normed space, a function $f:E\to F$ is called o-Lipschitz if there is an element L of F_{+} such that for every $x,y\in E$ we have:

$$-L\|x-y\|\leqq f\left(x\right)-f\left(y\right)\leqq L\|x-y\|.$$

Proposition 1. *Let E be a normed space. If $f:E_0\to F$ is locally o-Lipschitz, then f is m-continuous.*

Proof. Let $x^0\in E_0$, $V=B\left(x^0;r\right)$ be a ball of centre x^0 and radius $r\in\mathbb{R}_{+}^{*}$, on which f is o-Lipschitz, with Lipschitz constant $L\in F_{+}$. Then we have

$$-L\|x-y\|\leqq f\left(x\right)-f\left(y\right)\leqq L\|x-y\|,\ \forall x,y\in V.$$

Taking $y=x^0$, we have:

$$f\left(x^0\right)-L\|x-y\|\leqq f\left(x\right)\leqq f\left(x^0\right)+L\|x-y\|,\ \forall x\in V.$$

It follows that:

$$f\left(x^0\right)-Lr\leqq\sup_{x\in V}f\left(x\right)\leqq f\left(x^0\right)+Lr.$$

Hence $\tau-\lim_{V}\left(\sup h\left(V\right)\right)=f\left(x^0\right)$. In the same way we obtain that $\tau-\lim_{V}\left(\inf h\left(V\right)\right)=f\left(x^0\right)$, hence f is m-continuous.

Proposition 2. *Let $x^0\in E_0$ and $h:E_0\to F$ be a function which is m-continuous in x^0. Then h is τ-continuous.*

Proof. Let V_0 be a neighborhood of x^0 on which h is order bounded, and $V\subseteq V_0$ any other neighborhood. Denote $\inf h\left(V\right)$ with $i\left(V\right)$ and $\sup h\left(V\right)$ with $s\left(V\right)$. We have, for $x\in V$:

$$i\left(V\right)-s\left(V\right)\leqq h\left(x\right)-h\left(x^0\right)\leqq s\left(V\right)-i\left(V\right). \tag{1}$$

If U is a full neighborhood of 0, because the generalized sequences $i\left(V\right)$ and $s\left(V\right)$ are τ-convergent to $h\left(x^0\right)$ there is a neighborhood V_U of x^0 such that if $V\subseteq V_U$ then $\pm\left(s\left(V\right)-i\left(V\right)\right)\in U$. Because we can choose U full, it follows that the order interval $\left[\mathrm{i}(\mathrm{V})\text{-}\mathrm{s}(\mathrm{V}),\mathrm{s}(\mathrm{V})\text{-}\mathrm{i}(\mathrm{V})\right]$ is contained in U. Hence from (1) we have:

$$h(x)-h\left(x^{0}\right)\in U, \forall x\in V,$$

which means that h is τ-continuous in x^0.

We will use the following lemma (the proof is adapted from [3]):

Lemma 1. *Let F be a topological ordered vector space. If F_+ is closed, then every generalized sequence from F which is τ-convergent and increasing, has a least upper bound and this coincides with the topological limit of the sequence. It is also ω-convergent to this limit.*

Proof. Let $(x_\delta)_{\delta\in\Delta}$ be such a sequence, let l be his limit. If $\delta,\delta'\in\Delta$ $\delta\le\delta'$ then we have $x_\delta \leqq x'_\delta$ and taking the limit following δ' we obtain $x_\delta \leqq l$ because the cone F_+ is closed. Hence l is an upper bound of the sequence. If m is another upper bound, we have $x_\delta \leqq m$ for every $\delta\in\Delta$ hence taking the limit we obtain $l\leqq m$. Hence l is the least upper bound of the sequence.

Proposition 3. *Let $h: E_0 \to F$ be any function. We suppose that F_+ is closed. We consider the following two assertions:*

1) *h is m-continuous at $x^0 \in E_0$,*

2) $\underline{\lim}_{x\to x^0} h(x) = \overline{\lim}_{x\to x^0} h(x) = h\left(x^0\right)$.

Then:

a. 1) $\Rightarrow$ 2).
b. If the topology of F is ω continuous, then also 2) $\Rightarrow$ 1).

Proof. a. Because h is m-continuous, we have:

$$\tau-\lim_{V} i(\mathrm{V}) = h\left(x^0\right) = \tau-\lim_{V} \mathrm{s}(\mathrm{V}),$$

hence, F_+ being closed, we obtain with lemma 1:

$$\underline{\lim}_{x\to x^0} h(x) = \sup_{V\in\mathcal{V}x^0} i(\mathrm{V}) = \tau-\lim_{V\in\mathcal{V}x^0} \mathrm{s}(\mathrm{V}) = h\left(x^0\right).$$

In the same way, we have:

$$\overline{\lim_{x \to x^0}} h(x) = h(x^0).$$

b. The hypothesis means:

$$h(x^0) = \sup_{V \in \mathcal{V}x^0} i(\mathrm{V}) = \inf_{V \in \mathcal{V}x^0} \mathrm{s}(\mathrm{V}),$$

hence $i(\mathrm{V}) \uparrow h(x^0)$ and $s(\mathrm{V}) \downarrow h(x^0)$. This means $i(\mathrm{V}) \xrightarrow{\omega} h(x^0)$ and $s(\mathrm{V}) \xrightarrow{\omega} h(x^0)$ hence $i(\mathrm{V}) \xrightarrow{\tau} h(x^0)$ and $s(\mathrm{V}) \xrightarrow{\tau} h(x^0)$ because the topology of F is ω-continuous.

Hence h is m-continuous.

3. **Clarke's subdifferential**

This kind of continuous functions is useful to define Lipschitz functions and Clarke's derivative in spaces which are not metric spaces.

Definition 2. *A function $A : E \to F$ is called positive subhomogenuous, if*

$$A(\lambda) \leqq \lambda A(x),\ \forall \mathrm{x} \in \mathrm{E},\ \forall \lambda \in \mathbb{R},\ \lambda \geq 0.$$

Definition 3. *A function $f : E_0 \subseteq E \to F$ is called A-Lipschitz on E_0, if there is a positive subhomogenuous and m-continuous function $A : E \to F$, such that* $A(x) = A(-x),\ \forall \mathrm{x} \in \mathrm{E},$ *and*

$$-A(x-y) \leqq f(x) - f(y) \leqq A(x-y),\ \forall \mathrm{x,y} \in \mathrm{E}_0.$$

Definition 4. *Let $f : E_0 \subseteq E \to F$ be an A-Lipschitz function and let* $\mathrm{x}^0 \in \mathrm{E}_0$. *The Clarke's directional derivative of* f *in* x^0 *is the function* $f^0(x^0;.) : E \to F$ *defined by*

$$f^0(x^0; v) = \lim_{x \to x^0} \sup_{\lambda \searrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda}.$$

It is a good definition, because:

$$\pm \frac{f(x+\lambda v) - f(x)}{\lambda} \leqq \frac{A(\lambda v)}{\lambda} = A(v),$$

and $\limsup_{\substack{x\to x^0\\ \lambda\searrow 0}} \frac{f(x+\lambda v)-f(x)}{\lambda} \overset{def}{=} \inf_{\substack{V\in V_{x^0}\\ \varepsilon>0}} \sup_{\substack{x\in V\\ \lambda\in(0,\varepsilon)}} \frac{f(x+\lambda v)-f(x)}{\lambda}$ exists, because F is order complete.

Lemma 2. *Let* $f,g:E_0\subseteq E\to F$ *be two functions, order bounded in a neighborhood of* $a\in E_0$ *Then:*

$$\limsup_{x\to a}\big(f(x)+g(x)\big)\leqq \limsup_{x\to a} f(x)+\limsup_{x\to a} g(x).$$

Proof. Let $c\leqq \inf_{V\in V_a}\sup_{x\in V}\big(f(x)+g(x)\big)$ where $\mathcal{V}\in\mathcal{V}_a$ is the family of all neighborhoods of a. Let $V^0\in\mathcal{V}_a$ fixed. Then:

$$\inf_{V\in\mathcal{V}_a}\sup_{x\in V}\big(f(x)+g(x)\big)=\inf_{\substack{V\in\mathcal{V}_a\\ V\subseteq V^0}}\sup_{x\in V}\big(f(x)+g(x)\big),$$

because the generalized sequence $\left\{\sup_{x\in V}\big(f(x)+g(x)\big)\right\}_{V\in\mathcal{V}_a}$ is decreasing. We have:

$$c\leqq \sup_{x\in V}\big(f(x)+g(x)\big)\leqq \sup_{x\in V} f(x)+\sup_{x\in V} g(x),\ \forall \mathrm{V}\subseteq \mathrm{V}^0,\ \mathrm{V}\in\mathcal{V}_a,$$

and, successively:

$$c-\sup_{x\in V} g(x)\leqq \sup_{x\in V} f(x)$$

$$\inf_{V\subseteq V^0}\left(c-\sup_{x\in V} g(x)\right)\leqq \limsup_{x\to a} f(x)$$

$$c-\sup_{V\subseteq V^0}\left(\sup_{x\in V} g(x)\right)\leqq \limsup_{x\to a} f(x)$$

$$c-\limsup_{x\to a} f(x)\leqq \sup_{V\subseteq V^0}\left(\sup_{x\in V} g(x)\right)=\sup_{x\in V^0} g(x).$$

Because V^0 is arbitrarily, it follows:

$$c-\limsup_{x\to a} f(x)\leqq \inf_{V\in\mathcal{V}_a}\left(\sup_{x\in V} g(x)\right)=\limsup_{x\to a} g(x).$$

Hence, for every $c\in F$ for which $c\leqq \limsup_{x\to a}\big(f(x)+g(x)\big)$ we have:

$$c\leqq \limsup_{x\to a} f(x)+\limsup_{x\to a} g(x)$$

so

$$\limsup_{x\to a}\left(f(x)+g(x)\right)\leqq \limsup_{x\to a} f(x)+\limsup_{x\to a} g(x).$$

Lemma 3. *Let E be a topological vector space. The family of functions $\varphi_t : E\times E\to F$, $t\in[0,1]$, defined by:*

$$\varphi_t(x,y)=x+ty$$

is echicontinuous.

Proof. Let $\mathcal{V}_0$ an fundamental system of neighborhoods of 0 containing balanced neighborhoods. Then $\{V_0+ty_0\}_{V_0\in\mathcal{V}_0}$ is a fundamental system of neighborhoods of ty_0. If $t\in(0,1]$ we have:

$$t^{-1}(V_0+ty_0)=t^{-1}V_0+y_0\supseteq V_0+y_0,$$

because V_0 is balanced. It follows:

$$t(V_0+y_0)\subseteq t\left(t^{-1}V_0+y_0\right)=V_0+ty_0, \forall V_0\in\mathcal{V}_0, \forall t\geq 0,$$

because for $t=0$ the inclusion is obvious. Let $U\in\mathcal{V}_0$ i.*e.* $U+U\subseteq V_0$. Then we have:

$$\varphi_t(U+x_0,U+y_0)=U+x_0+t(U+y_0)\subseteq U+x_0+U+ty_0$$
$$\subseteq V_0+x_0+ty_0$$

hence $\{\varphi_t(\cdot,\cdot)\}_{t\in[0,1]}$ is equicontinuous.

Theorem 1. *Let $f: E_0\subseteq E\to F$ be a function locally A-Lipschitz on E_0. Then:*

1. $f^0(x^0;.)$ *is positive homogeneous and subaditive on E, and we have:*

$$\pm f^0(x,v)\leqq A(v)$$

where A is the Lipschitz constant of f.

2. The function $(x,v)\mapsto f^0(x,v)$ is upper semicontinuous.

3. The function $f^0(x^0;.)$ is A-Lipschitz on E (with Lipschitz constant A).

4. $f^0(x,-v)=(-f)^0(x,v)$.

Proof. 1. Clearly, $\pm f^0(x,v)\leqq A(v)$ and $f^0(x,\lambda v)=\lambda f^0(x,v)$, if $\lambda\geq 0$. We have:

$$f^0(x;v+w)=$$
$$\limsup_{\substack{y\to x\\ t\searrow 0}}\frac{f(y+tv+tw)-f(y)}{t}\le \limsup_{\substack{y\to x\\ t\searrow 0}}\frac{f(y+tv+tw)-f(y+tw)}{t}+$$
$$+\limsup_{\substack{y\to x\\ t\searrow 0}}\frac{f(y+tw)-f(y)}{t}=f^0(x;v)+f^0(x;w)$$

hence $f^0(x,.)$ is subaditive.

2. Let $x^0\in E_0$, $v^0\in E$, $V_0\in \mathcal{V}_{x^0}$ be arbitrarily. With Lemma 2, let $U_1\in\mathcal{V}_{x^0}$, $U_1\subseteq V_0$, $U_2\in\mathcal{V}_{x^0}$, be such that:

$$U_1+tU_2\subseteq V_0+tv^0. \tag{2}$$

(2) Let $t\neq 0$ and $y\in U_1$, be arbitrarily, and $x_i\in E_0$, $x_i\to x^0$, $v_i\in E$, $v_i\to v^0$, $i\in N$. Let $i_0\in N$ be such that $i\ge i_0\Rightarrow x_i\in U_1$ and $v_i\in U_2$. Then from (2), it follows $y+tv_i=y'+tv_0$ with $y'\in V_0$. Hence we have:

$$\frac{f(y+tv_i)-f(y)}{t}=\frac{f(y'+tv^0)-f(y')}{t}+\frac{f(y')-f(y)}{t}.$$

But $y'=y+t(v_i-v^0)$, hence we obtain:

$$\frac{f(y+tv_i)-f(y)}{t}=\frac{f(y'+tv^0)-f(y')}{t}+\frac{f\left(y+t\left(v_i-v^0\right)\right)-f(y)}{t}.$$

Taking the least upper bound following $y\in U_1$, we have:

$$\sup_{y\in U_1}\frac{f(y+tv_i)-f(y)}{t}\le\sup_{y'\in V_0}\frac{f(y'+tv^0)-f(y')}{t}+$$
$$+\sup_{y\in V_0}\frac{f\left(y+t\left(v_i-v^0\right)\right)-f(y)}{t}\le\sup_{y'\in V_0}\frac{f(y'+tv^0)-f(y')}{t}+$$
$$+A\left(v_i-v^0\right).$$

Taking now the supremum following $t\in(0,\varepsilon]$, we have:

$$\sup_{\substack{y\in U_1\\ 0<t\le\varepsilon}}\frac{f(y+tv_i)-f(y)}{t}\le\sup_{\substack{y'\in V_0\\ 0<t\le\varepsilon}}\frac{f(y'+tv^0)-f(y')}{t}+A\left(v_i-v^0\right).$$

On the other hand,

$$f^0\left(x_i, v_i\right) = \inf_{\substack{V \in \mathcal{V}_{x_i} \\ \varepsilon > 0}} \sup_{\substack{y \in V \\ t \in (0,\varepsilon)}} \frac{f\left(y + tv_i\right) - f\left(y\right)}{t} \leqq$$

$$\leqq \inf_{\substack{V \subseteq U_1 \\ \varepsilon > 0}} \sup_{\substack{y \in V \\ t \in (0,\varepsilon)}} \frac{f\left(y + tv_i\right) - f\left(y\right)}{t} \leqq \sup_{\substack{y \in U_1 \\ t \in (0,\varepsilon)}} \frac{f\left(y + tv_i\right) - f\left(y\right)}{t}$$

for every $i \geq i_0$. Hence, for every $V_0 \in \mathcal{V}_{x^0}$ there is an $i_0 \in \mathbb{N}$, such that $i \geq i_0$ implies:

$$f^0\left(x_i; v_i\right) \leqq \sup_{\substack{y' \in V_0 \\ t \in (0,\varepsilon]}} \frac{f\left(y' + tv^0\right) - f\left(y'\right)}{t} + A\left(v_i - v^0\right).$$

We take first least upper bound following $i \geq n$ and we obtain:

$$\sup_{i \geq n} f^0\left(x_i; v_i\right) \leqq \sup_{\substack{y' \in V_0 \\ t \in (0,\varepsilon]}} \frac{f\left(y' + tv^0\right) - f\left(y'\right)}{t} + \sup_{i \geq n} A\left(v_i - v^0\right).$$

Now we take greatest lower bound following $n \in \mathbb{N}$:

$$\inf_{n \in \mathbb{N}} \sup_{i \geq n} f^0\left(x_i; v_i\right) \leqq \sup_{\substack{y' \in V_0 \\ t \in (0,\varepsilon]}} \frac{f\left(y' + tv^0\right) - f\left(y'\right)}{t} + \inf_{n \in \mathbb{N}} \sup_{i \geq n} A\left(v_i - v^0\right),$$

And, finally, greatest lower bound following V_0 and ε:

$$\inf_{n \in \mathbb{N}} \sup_{i \geq n} f^0\left(x_i; v_i\right) \leqq \inf_{\substack{V_0 \in \mathcal{V}_{x^0} \\ \varepsilon > 0}} \sup_{\substack{y' \in V_0 \\ t \in (0,\varepsilon]}} \frac{f\left(y' + tv^0\right) - f\left(y'\right)}{t} + \inf_{n \in \mathbb{N}} \sup_{i \geq n} A\left(v_i - v^0\right).$$

Hence we have the relations:

$$\limsup_{i \to \infty} f^0\left(x_i; v_i\right) \leqq \inf_{\substack{V_0 \in \mathcal{V}_{x^0} \\ \varepsilon > 0}} \sup_{\substack{y' \in V_0 \\ t \in (0,\in]}} \frac{f\left(y' + tv^0\right) - f\left(y'\right)}{t} + \limsup_{i \to \infty} A\left(v_i - v^0\right) =$$

$$= f^0\left(x^0, v^0\right),$$

because A being m-continuous, we have:

$$\limsup_{i \to \infty} A\left(v_i - v^0\right) = 0.$$

3. The Lipschitz condition implies:

$$-A\left(t\left(v - w\right)\right) + f\left(y + tw\right) \leqq f\left(y + tw\right) \leqq f\left(y + tw\right) + A\left(t\left(v - w\right)\right)$$

which means:

$$-A(v-w)+\frac{f(y+tw)-f(y)}{t} \leqq \frac{f(y+tw)-f(y)}{t} \leqq$$
$$\leqq \frac{f(y+tw)-f(y)}{t} + A(v-w).$$

Taking $\lim_{V \in \mathcal{V}_x} \sup_{\varepsilon > 0}$, we obtain:

$$-A(v-w)+f^0(x;w) \leqq f^0(x;v) \leqq f^0(x;w)+A(v-w),$$

which is equivalent to:

$$\pm\left(f^0(x;v)-f^0(x;w)\right) \leqq A(v-w),$$

hence 3.

4. We have, using the notation $u = x' + tv$:

$$f^0(x;-v) \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{f(x'-tv)-f(x')}{t} = \limsup_{\substack{u \to x \\ t \searrow 0}} \frac{f(u+tv)-(-f)(u)}{t} =$$
$$=(-f)^0(x;v).$$

Lemma 4 (see [4]). *If F is an order complete vector lattice, E_0 a subspace of E, $p: E \to F$ a sublinear function, and $U_0 : E_0 \to F$ is a linear operator upper bounded by p, then there is a linear extension U of U_0 which is also upper bounded by p.*

Corollary 1. *In the context of the previous lemma, if $p: E \to F$ is a sublinear function, then for every $v_0 \in E$ there is a linear operator $U : E \to F$ such that*

$$U(v_0) = p(v_0) \text{ and } U(v) \leqq p(v), \ \forall \mathrm{v} \in \mathrm{E}.$$

Proof. We take $E_0 = \{\lambda v_0 : \lambda \in \mathbb{R}\}$ and we define $U_0 : E_0 \to F$ by

$$U(\lambda v_0) = \lambda p(v_0).$$

Because p is subaditive and positive – homogeneous, we apply the previous lemma. The operator U has also the property $U(v_0) = p(v_0)$.

Definition 5. *The subgradient of $f : E_0 \subseteq E \to F$ at $x^0 \in E_0$ is the set*

$$\partial f(x^0) = \left\{U \in \pounds(\mathrm{E,F}) : \ \mathrm{U}(\mathrm{v}) \leqq f(x^0;v), \ \forall \mathrm{v} \in \mathrm{E}\right\}.$$

Theorem 2. *We suppose that the conditions of theorem [1] are satisfied. Then:*

1. *The set* $\partial f(x)$ *is non-empty and convex.*

2. *Every* $U \in \partial f(x)$ *is m-continuous and we have*

$$\pm U(v) \leqq A(v),\ \forall \mathrm{v} \in \mathrm{E}.$$

3. $\forall \mathrm{v} \in \mathrm{E},\quad f^0(x;v) = \max\{\ \mathrm{U}(\mathrm{v}) : U \in \partial f(x)\ \}.$

Proof. 1. The function $f^0(x;.)$ is sublinear. From Corollary (1), $\partial f(x) \neq \varnothing$. We have:

$$U(v) \leqq f^0(x;v) \leqq A(v),\ \forall \mathrm{v} \in \mathrm{E},$$

hence:

$$-U(v) = U(-v) \leqq f^0(x;-v) \leqq A(-v) = A(v),\ \forall \mathrm{v} \in \mathrm{E},$$

thus:

$$\pm U(v) \leqq A(v),\ \forall \mathrm{v} \in \mathrm{E}.$$

Because A is m-continuous, it follows that U is m-continuous.

2. Let $\mathrm{v}_0 \in \mathrm{E}$. From Corollary (1) it follows that there is $U_0 : \mathrm{E} \to \mathrm{F}$ a linear operator, such that:

$$U_0(v) \leqq f^0(x;v),\ \forall \mathrm{v} \in \mathrm{E}, \text{ and } U_0(v_0) = f^0(x;v_0).$$

Hence $U_0 \in \partial f(x)$ and $f^0(x;v_0) = \max\{\ \mathrm{U}(\mathrm{v}_0) : U \in \partial f(x)\ \}$.

We will use next the following result.

Proposition 4 (see [1]). *Let* E, F *be two topological vector spaces,* F *a separate one. A subset* $\mathcal{H}$ *of* $\mathcal{L}(E, F)$ *is relatively compact in* $\mathcal{L}(E, F)$*, endowed with the simple convergence topology, if and only if for every* $x \in E$ *the set:*

$$\mathcal{H}(x) = \{H(x) : H \in \mathcal{H}\}$$

is relatively compact in F.

Proposition 5. *We suppose that the conditions of theorem [1] are satisfied, and that the topology of* F *is generated by a norm* $\|\cdot\|$.

If the norm is monotone, which means:

$$\pm x \leqq y \Rightarrow \|x\| \leq \|y\|$$

then:

1. $\partial f(x) \in \mathcal{L}(E,F)$ is echicontinuous.

2. If for every $a,b \in F$ the intervals $[a,b]$ are σ-*compact*, then $\partial f(x)$ is σ-*compact* in $\mathcal{L}_s(E, F_\sigma)$ where F_σ is endowed with the weak topology $\sigma(E, F^*)$, and in $\mathcal{L}(E, F_\sigma)$ is considered the topology of simple convergence.

Proof. 1. For every $U \in \partial f(x)$ we have $\pm U(v) \leqq A(v)$, $\forall v \in E$ hence by the monotonicity of norm, it follows that $\|U(v)\| \leq \|A(v)\|$, hence the echicontinuity of the family $\partial f(x)$.

2. We show that $\partial f(x)$ is echicontinuous in $\mathcal{L}_s(E, F_\sigma)$ too. Indeed, for every $E_0 \subseteq F_\sigma$ open and containing 0, there is $G \subseteq F$ open neighborhood of 0 such that $G \subseteq E_0$. Then there is V neighborhood of 0 in E such that $U(V) \subseteq G \subseteq E_0$, for every $U \in \partial f(x)$, because $\partial f(x)$ is echicontinuous in $\mathcal{L}(E, F)$. It follows that $\partial f(x)$ is echicontinuous in $\mathcal{L}_s(E, F_\sigma)$ too.

We have $-A(v) \leqq U(v) \leqq A(v)$, hence the orbit $\{U(v) : U \in \partial f(x)\}$ is relatively σ-*compact*, because the intervals are relatively σ-*compact*. It follows from Proposition (4) that $\partial f(x)$ is relatively σ-*compact* in $\mathcal{L}_s(E, F_\sigma)$. On the other hand, $\partial f(x)$ is closed in $\mathcal{L}_s(E, F_\sigma)$. Indeed, let $(U_\delta)_{\delta \in \Delta}$ be a generalized sequence such that U_δ is pointwise convergent to U. Let $\alpha \in F^*, \alpha \geq 0$ we have:

$$\langle \alpha; U_\delta(v) \rangle \leq \langle \alpha; f^0(x; v) \rangle, \ \forall v \in E.$$

Taking the limit, we obtain:

$$\langle \alpha; U(v) \rangle \leq \langle \alpha; f^0(x; v) \rangle, \ \forall v \in E.$$

Because the positive cone of F is closed and α is arbitrarily, it follows that ([3], prop. 1, pp. 92):

$$U(v) \leqq f^0(x; v), \ \forall v \in E,$$

hence $U \in \partial f(x)$. Thus $\partial f(x)$ is closed in $\mathcal{L}_s(E, F_\sigma)$, hence compact.

4. **Relation with other derivatives**

Definition 6. *A normed lattice of type (M) is a normed lattice F in which the norm satisfy the condition:*

$$0 \leqq x, y \Rightarrow \|x \vee y\| = \max\{\|x\|, \|y\|\}, \ \forall x, y \in F.$$

Definition 7. *A normed lattice of type (L) is a normed lattice* F *in which the norm satisfy the condition:*

$$0 \leqq x, y \Rightarrow \|x + y\| = \|x\| + \|y\|.$$

The link between these two types is showed in the following well-known proposition.

Proposition 6. *If* F *is a normed lattice of type (M), then* F_τ^* *is a normed lattice of type (L).*

Theorem 3. *Let* $f : E_0 \subseteq E \to F$ *a function,* E_0 *a neighborhood of* x *in the Banach space* E*, and* F *a separable Banach lattice of type (M). The following conditions are equivalent:*

i) f is strictly-differentiable at x, with the strict derivative denoted U*;*

ii) f is o-Lipschitz around x and for every $v \in E$ the limit

$$\tau - \lim_{\substack{x' \to x \\ t \searrow 0}} \frac{f(x'+tv) - f(x)}{t} \overset{not}{=} U(v)$$

exists.

Remark 1. *The implication* $ii) \Rightarrow i)$ *is true without the condition* F *separable and of type (M).*

Proof. $i) \Rightarrow ii)$ We suppose that $i)$ is true. Then it is known that the equality stated by $ii)$ take place. We show next that f is o-Lipschitz around x. If it is not the case, for every neighborhood V of x, and for every $p \in F_+$, $p \geq 0$ $p \neq 0$ there is $x', x'' \in V$, such that:

$$|f(x') - f(x'')| \nleqq p.\|x' - x''\|.$$

Let $V_n = B\left(x; \frac{1}{n}\right)$, $p_n = np_0$ with $p \in F_+ \setminus \{0\}$. Then there is x_n', x_n'' two sequences which are convergent to x, such that:

$$|f(x_n') - f(x_n'')| \nleqq np_0.\|x_n' - x_n''\|, \ \forall \mathrm{n} \in \mathbb{N},$$

which means:

$$|f(x_n') - f(x_n'')| \notin np_0.\|x_n' - x_n''\| - F_+, \ \forall \mathrm{n} \in \mathbb{N}.$$

Because the positive cone F_+ is closed, the set $np_0.\|x_n' - x_n''\| - F_+$ is closed and convex, hence with the separation theorem, there is $\alpha_\mathrm{n} \in F_\tau^*$ such that $\|\alpha_\mathrm{n}\| = 1$ and

$$\langle \alpha_n; |f(x_n') - f(x_n'')| \rangle \geq \langle \alpha_n; y \rangle, \ \forall \mathrm{y} \leqq np_0.\|x_n' - x_n''\|, \ \forall \mathrm{n} \in \mathbb{N}.$$

It follows that $\alpha_\mathrm{n} \geq 0$ and, in particular,

$$\langle \alpha_n; |f(x_n') - f(x_n'')| \rangle \geq \langle \alpha_n; p_0 \rangle .n.\|x_n' - x_n''\|, \ \forall \mathrm{n} \in \mathbb{N}. \tag{3}$$

Let $t_\mathrm{n} > 0$ and $v_\mathrm{n} \in E$, be determined by the condition:

$$t_n v_n = x_n'' - x_n', \|v_n\| = \frac{1}{\sqrt{n}}, \forall n \in \mathbb{N}.$$

Then we have:

$$t_n = \frac{\|x_n'' - x_n'\|}{v_n} = \sqrt{n}\,\|x_n'' - x_n'\| \le \sqrt{n}.\frac{2}{n} \to 0.$$

From (3) it follows:

$$\left\langle \alpha_n; \left| f\left(x_n' + t_n v_n\right) - f\left(x_n'\right) \right| \right\rangle \ge t_n \left\langle \alpha_n; p_0 \right\rangle .n.\|v_n\|,\ \forall \mathrm{n} \in \mathbb{N},$$

and so:

$$\left\langle \alpha_n; \frac{\left| f\left(x_n' + t_n v_n\right) - f\left(x_n'\right) \right|}{t_n} \right\rangle \ge \left\langle \alpha_n; p_0 \right\rangle .\sqrt{n},\ \forall \mathrm{n} \in \mathbb{N}. \tag{4}$$

Because the space F_τ^* is of type (L), the set:

$$S_+ = \left\{ \alpha \in F_\tau^* : \alpha \ge 0,\ \text{and } \|\alpha\| = 1 \right\}$$

is convex, hence σ^* closed, thus σ^* is compact by Alaoglu's theorem. Because F is separable, the closed unit ball in the dual F_τ^*

$$B^*\left(0,1\right) = \left\{ \alpha \in F_\tau^* :\ \|\alpha\| \le 1 \right\}$$

endowed with the topology σ^* is a metrizable space. We can hence suppose that $\left(\alpha_n\right)_n$ is σ^*-convergent to some $\alpha^0 \in S_+$. If $\lim_n \left\langle \alpha_n; p_0 \right\rangle = 0$, then $\left\langle \alpha^0; p_0 \right\rangle = 0$, hence p_0 is a support point for F_+. But because the subspace generated by F_+ is F, and F is separable, there is a point of F_+ which is not a support point for F_+. We suppose that p_0 is such a point. Then $\lim_n \left\langle \alpha_n; p_0 \right\rangle \ne 0$. Hence, for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that $\left\langle \alpha_n; p_0 \right\rangle \ge \varepsilon \ \ \forall n \ge n_\varepsilon$. From (4), we obtain:

$$\left\langle \alpha_n; \frac{\left| f\left(x_n' + t_n v_n\right) - f\left(x_n'\right) \right|}{t_n} \right\rangle \ge \varepsilon.\sqrt{n},\ \forall \mathrm{n} \in \mathbb{N}, n \ge n_\varepsilon. \tag{5}$$

On the other hand, the set $K = \left\{ v_n : n \in \mathbb{N} \right\} \cup \left\{ 0 \right\}$ is compact in E and from the definition of the strict derivative, it follows that for every $\varepsilon' > 0$ there is $n_{\varepsilon'} \in \mathbb{N}$ such that for every $\forall n \ge n_{\varepsilon'}$, we have:

$$\left\| \frac{f\left(x_n' + t_n v_n\right) - f\left(x_n'\right)}{t_n} - U\left(v\right) \right\| \le \varepsilon',\ \forall \mathrm{v} \in \mathrm{K}.$$

Taking $v = v_n$ and $\varepsilon' = \varepsilon$ we obtain:

$$\left\| \frac{f(x_n' + t_n v_n) - f(x_n')}{t_n} \right\| \le \|U(v_n)\| + \varepsilon \le \|U\| + \varepsilon .$$

Hence the sequence $\left(\frac{f(x_n' + t_n v_n) - f(x_n')}{t_n} \right)_{n \in \mathbb{N}}$ is bounded. Also, the sequence $(\alpha_n)_n$ is bounded too. But because the bilinear mapping $(\alpha, y) \mapsto \langle \alpha, y \rangle$ from $F_\tau^* \times F$ to $\mathbb{R}$ is ($\sigma^* \times \tau$)-continuous, this contradicts (5). Hence f is 0-Lipschitz around x.

$ii) \Rightarrow i)$ It is known that if the equality from $ii)$ is true, and f is locally Lipschitz in x, then f is strictly differentiable in x. Let $L \in F$ be the Lipschitz constant of f. Because the norm of F is monotone, for a neighborhood V of x we have:

$$\|f(x) - f(y)\| \le L . \|x - y\|$$

Hence, f is locally Lipschitz at x, hence it is strictly-differentiable at x, and U is the strict derivative of f at x.

Proposition 7. *Let E be a topological vector space, F a topological ordered vector space, which is order complete. Let $f : E_0 \subseteq E \to F$ be an A-Lipschitz function in a neighborhood of $x \in \operatorname{int} E_0$ having a Gateaux derivative $D_g f(x)$*

Then $D_g f(x) \in \partial f(x)$.

Proof. By hypothesis, there exists $D_g f(x) : E \to F$ a linear operator such that:

$$f'(x; v) = \lim_{t \searrow 0} \frac{f(x + tv) - f(x)}{t} = D_g f(x)(v).$$

We have, for every $V \in \mathcal{V}_x$ and every $\varepsilon > 0$:

$$\frac{f(x + tv) - f(x)}{t} \leqq \sup_{\substack{y \in V \\ s \in (0, \varepsilon)}} \frac{f(y + sv) - f(y)}{s}, \ \forall t \in (0, \varepsilon),$$

Hence:

$$\lim_{t \searrow 0} \frac{f(x + tv) - f(x)}{t} \leqq \sup_{\substack{y \in V \\ s \in (0, \varepsilon)}} \frac{f(y + sv) - f(y)}{s}, \ \forall V \in \mathcal{V}_x, \ \forall \varepsilon > 0 .$$

Finally,

$$\lim_{t\searrow 0}\frac{f(x+tv)-f(x)}{t} \leqq \inf_{\substack{V\in V_x\\ \varepsilon>0}} \sup_{\substack{y\in V\\ s\in(0,\varepsilon)}} \frac{f(y+sv)-f(y)}{s},$$

which means $D_g f(x)\in\partial f(x)$.

Corollary 2. *In this context, because the Frechet derivative* $Df(x)$ *and the strict derivative* $D_s f(x)$ *are Gateaux derivatives too, we have* $Df(x)\in\partial f(x)$ *and* $D_s f(x)\in\partial f(x)$.

Lemma 5. *Let* F *be a normed lattice in which the unit ball is order bounded. Let* $x\in F$ *be arbitrarily. If we denote* $i_n=\inf B\left(x;\frac{1}{n}\right)$, *and* $s_n=\sup B\left(x;\frac{1}{n}\right)$ *then we have:*

$$\sup_n i_n = x = \inf_n s_n .$$

Proof. Denote $B_n=B\left(x;\frac{1}{n}\right)$. Let u be an upper bound for B_1. Because $B_n\subseteq B_{n-1}$ we have $x'\leqq u,\ \forall x'\in B_n$. If $y\in B_n$, then:

$$\|y-x\|\le\frac{1}{n}\Rightarrow\|ny-nx\|\le 1\Rightarrow\|ny-(n-1)x-x\|\le 1,$$

Hence, $ny-(n-1)x\in B_1$, thus we have $ny-(n-1)x\leqq u$, which means $y\leqq\frac{1}{n}x+\frac{1}{n}u+x$. So:

$$x\leqq\sup B_n\leqq\frac{1}{n}(x+u)+x,\ \forall n\in\mathbb{N}^*,$$

and we have from this:

$$x=\inf_n\sup B_n=\inf_n s_n$$

because a normed lattice is an archimedian space.

Analogously, we obtain $\sup_n\inf B_n=\sup_n s_n=x$.

Proposition 8. *We suppose that the topology of* F *is defined by a norm and that the unit ball is order bounded. If* $f:E_0\subseteq E\to F$ *is strictly-differentiable at* x

and it is A-Lipschitz around x*, then* $\partial f(x) = \{D_s f(x)\}$ *where* $D_s f(x)$ *is the strict derivative of* f *at* x.

Proof. By proposition (7), $D_s f(x) \in \partial f(x)$. Conversely, we show that:

$$f^0(x;v) = D_s f(x)(v)$$

For every n there is a neighborhood V_n of x and there is $\varepsilon_n > 0$ such that:

$$\frac{f(x'+\lambda v) - f(x')}{\lambda} \in D_s f(x)(v) + B\left(0;\frac{1}{n}\right),\ \forall \mathrm{x'} \in \mathrm{V_n},\ \forall \lambda \in (0,\varepsilon_{\mathrm{n}}).$$

It follows that:

$$\sup_{\substack{\mathrm{x'}\in \mathrm{V_n} \\ \lambda \in (0,\varepsilon_{\mathrm{n}})}} \frac{f(x'+\lambda v) - f(x')}{\lambda} \le \sup\left(D_s f(x)(v) + B\left(0;\frac{1}{n}\right)\right) .$$

We have:

$$f^0(x;v) = \inf_{\substack{V \in \mathcal{V}_x \\ \varepsilon > 0}} \sup_{\substack{\mathrm{x'}\in \mathrm{V_n} \\ \lambda \in (0,\varepsilon_{\mathrm{n}})}} \frac{f(x'+\lambda v) - f(x')}{\lambda} \leqq \inf_n \sup_{\substack{\mathrm{x'}\in \mathrm{V_n} \\ \lambda \in (0,\varepsilon_{\mathrm{n}})}} \frac{f(x'+\lambda v) - f(x')}{\lambda} \leqq$$

$$\leqq \inf_n \sup\left(D_s f(x)(v) + B\left(0;\frac{1}{n}\right)\right) = D_s f(x)(v) + \inf_n \sup B\left(0;\frac{1}{n}\right) =$$

$$= D_s f(x)(v)$$

Hence, $f^0(x;v) \leqq D_s f(x)(v)$. The converse inequality is true, hence the wanted equality.

Proposition 9. *We suppose that the topology of* F *is* ω*-continuous. If* f *is A-Lipschitz around* x *and* $\partial f(x) = \{U\}$*,* $U \in \mathcal{L}(E,F)$ *then* f *is strictly-differentiable in* x *and* $D_s f(x) = U$.

Proof. We show that $f^0(x;v) = U(v),\ \forall \mathrm{v} \in \mathrm{E}$. We have $f^0(x;v) \ge U(v),\ \forall \mathrm{v} \in \mathrm{E}$. If there is v' such that $f^0(x;v') \ngeqq U(v'),\ \forall \mathrm{v} \in \mathrm{E}$ then from theorem (1), there is $U': E \to F$ a continuous linear operator such that:

$$f^0(x;v) \geqq U'(v),\ \forall \mathrm{v} \in \mathrm{E},$$

and $f^0(x;v)=U'(v')$. Hence $U'\in\partial f(x)$. Because $U'(v')\neq U(v')$ we have $\partial f(x)\neq\{U\}$, a contradiction. Hence $f^0(x;.)=U$ and $f^0(x;-v)=-f^0(x;v)$.

We take now a generalized sequence $(x_V)_{V\in\mathcal{V}}$ indexed with the neighborhoods of x, such that $x_V\in V$. Then the sequence is τ - convergent to x. Also, we take a generalized sequence $(t_V)_{V\in\mathcal{V}}$ of real numbers, decreasing to 0.

$$\frac{f(x_v+t_v v)-f(x_v)}{t_v}\leqq\sup_{\substack{y\in V\\ \lambda\in(0,t_v)}}\frac{f(y+\lambda v)-f(y)}{\lambda}.$$

The right part is a generalized sequence, ω-convergent to $f^0(x;v)$. Because the topology of F is ω-continuous, it is also τ - convergent to $f^0(x;v)=U(v)$. Hence we have:

$$\lim_{V\in\mathcal{V}}\frac{f(x_v+t_v v)-f(x_v)}{t_v}\leqq U(v).$$

On the other hand, because $f^0(x;-v)=-f^0(x;v)$ we have:

$$f^0(x;v)=\sup_{\substack{V\in\mathcal{V}_x\\ \varepsilon>0}}\inf_{\substack{x'\in V\\ \lambda\in(0,\varepsilon)}}\left(-\frac{f(x'-\lambda v)-f(x')}{\lambda}\right)=\sup_{\substack{W\in\mathcal{V}_x\\ \varepsilon>0}}\inf_{\substack{y\in W\\ \lambda\in(0,\varepsilon)}}\frac{f(y+\lambda v)-f(y)}{\lambda},$$

where we have denoted $x'-\lambda v=y$. Also we have:

$$\frac{f(x_v+t_v v)-f(x_v)}{t_v}\geqq\inf_{\substack{y\in V\\ \lambda\in(0,t_v)}}\frac{f(y+\lambda v)-f(y)}{\lambda}.$$

The right part is ω-convergent to $f^0(x;v)=U(v)$, hence τ-convergent, because the topology of F is ω-continuous. Taking the limit, we have:

$$\lim_{V\in\mathcal{V}}\frac{f(x_v+t_v v)-f(x_v)}{t_v}=U(v),\forall x_n\to x,\forall t_n\downarrow 0,$$

hence:

$$\sup_{\substack{x'\to 0\\ t\searrow 0}}\frac{f(x'+tv)-f(x')}{t}=U(v),\ \forall \mathrm{v}\in\mathrm{E}.$$

Now from theorem (3) it follows that f is strictly differentiable in x and $D_s f(x)=U$.

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