A FORMULA CONCERNING SYMMETRIC BILINEAR MAPS AND SELF ADJOINT OPERATORS

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Abstract

We prove a formula giving a limit in which continuous bilinear and symmetric functionals appear. Alternatively, the formula is expressed in terms of self adjoint operators.

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1. Introduction

We prove a formula involving a symmetric and continuous bilinear functional. Due to the correspondence between this type of functionals and the self adjoint operators, the same formula will be expressed in terms of self adjoint operators. For general differentiability theory, we refer to [1] and [3]. For functional analysis notions, we refer to [2] and [3].

2. Results

A. We begin with a preliminary result.

Let E, F be normed spaces (over $\mathbb{R}$ or $\mathbb{C}$).

Let also $T : E \times E \to F$ be a bilinear, symmetric and continuous map. We define the function $f : E \to F$ via

$$f(x) = \frac{1}{2} T(x, x)$$

Lemma. The function f is differentiable of class $\mathbb{C}^\infty$. More precisely:

For any $a \in E$, one has

$$d^2 f(a) = T$$

Consequently, for any $a \in E$ and any $n \geq 3$, one has

$$d^n f(a) = 0$$

Proof (We give the proof for the sake of completeness).

One has the obvious equality

$$f = \frac{1}{2} T \circ h,$$

where $h : E \to E \times E$ is the linear and continuous map given via

$$h(x) = (x, x).$$

Hence, for any $u \in E$

$$df(u) = \frac{1}{2} dT(h(u)) \circ h = \frac{1}{2} dT(u,u) \circ h \tag{1}$$

It is known that, for all a, b in E and all x, y in E, one has

$$dT(a,b)(x,y) = T(x,b) + T(a,y)$$

hence, for all u, b in E, (1) gives

$$df(u)(b) = \frac{1}{2} dT(u,u)(h(b)) = \frac{1}{2} dT(u,u)(b,b) = \frac{1}{2}(T(b,u) + T(u,b)) = T(u,b) \tag{2}$$

Implicitely, we made use of the fact that T is of class $\mathbb{C}^\infty$.

Now, we have the function

$\varphi : E \to L(E,F) = \{V : E \to F \mid V \text{ is linear and continuous } \}$,

given via

$$\varphi(u) = df(u) \tag{*}$$

and we must prove that

$$d\varphi(u) = T \tag{3}$$

for all $u \in E$, using the canonical identification between *L*(*E*, *L*(*E,F*)) and $L(E,F) = \{H : E \times E \to F \mid H \text{ is linear and continuous}\}$. Recall that the identification is the following:

$$\mathrm{V} \in \mathrm{L}(\mathrm{L}(\mathrm{E}, \mathrm{L}(\mathrm{E}, \mathrm{F})), \mathrm{H} \in \mathrm{B}(\mathrm{E}, \mathrm{F}) \to \mathrm{V} \equiv \mathrm{H}$$

Namely, if V is given, we have, for all u, b in E, $H(u,b) = V(u)(b)$

Consequently, in order to prove (3), one must prove that, in the normed space L(*E,F*) one has for all $u \in E$:

$$\lim_{x \to u} \frac{1}{\|x - u\|} (\varphi(x) - \varphi(u) - V(x-u)) = 0 \tag{4}$$

Here, for all $b \in E$, one has $V(b) = T_b$, where $T_b \in$ L(*E,F*) is defined via

$$T_b = T(b,c) \tag{5}$$

for all $c \in E$.

But, using (*) and (2), we have, for all u, b in E

$$\varphi(u)(b) = T(u,b) \tag{6}$$

Relations (5) and (6) make (4) obvious: for any $x,u,y \in E$, one has

$$(\varphi(x) - \varphi(u) - V(x-u))(y) = \varphi(x)(y) - \varphi(u)(y) - V(x-u)(y) =$$
$$T(x,y) - T(u,y) - T_{x-u}(y) = T(x,y) - T(u,y) - T(x-u,y) = 0.$$

B. We shall use the previous lemma for a real Hilbert space E, equipped with the scalar product $(x,y) \to (x \mid y)$) and for $F = \mathbb{R}$.

Proposition 1. Let E be a real Hilbert space and $a \in E$. Let $T : E \times E \to \mathbb{R}$ be a bilinear, symmetric and continuous map (functional).

Then, there exist two continuous functions $\omega : E \to \mathbb{R}$, $\varepsilon : E \to E$, such that $\omega(a) = 0, \varepsilon(a) = 0$ and having the property that, for all $x \in E$, one has

$$\lim_{t \to 0} \frac{1}{\|t\|}\left(T(x,t) + \|x\|\omega(a+x) - \|x+t\|\omega(a+x+t) + \|x\|(\varepsilon(a+x) \mid t)\right) = 0$$

Proof. Let us define $f : E \to \mathbb{R}$ via

$$f(x) = \frac{1}{2}T(x,x)$$

Apply the lemma to see that, for any $b \in F$

$$d^2 f(b) = T \tag{**}$$

Of course, f and df are continuous.

Let $a \in E$ be arbitrarily taken. Because f is differentiable at a, we find a function $\omega : E \to \mathbb{R}$ which is continuous and has the property $\omega(a) = 0$, such that, for all u,v in E one has

$$f(v) - f(a) = df(a)(v-a) + \|v-a\|\omega(v)$$
$$f(u) - f(a) = df(a)(u-a) + \|u-a\|\omega(u)$$

This implies that, for any u,v in E one has

$$f(v) - f(u) = df(a)(v-u) + \|v-a\|\omega(v) - \|u-a\|\omega(u) \tag{7}$$

Now, let us fix $u \in E$ arbitrarily. Because f is differentiable at u, we can find a continuous function $\Omega : E \to \mathbb{R}$, such that $\Omega(u) = 0$, having the property that, for all v in E, one has

$$f(v) - f(u) = df(u)(v-u) + \|v-u\|\Omega(v) \tag{8}$$

The function $df : E \to E'$ is differentiable at a (here E' is the dual of E). We can find a continuous function $\varepsilon_1 : E \to E'$, such that $\varepsilon_1(a) = 0$ and for all $u \in E$

$$df(u) = df(a) + d^2 f(a)(u-a) + \|u-a\|\varepsilon_1(u) \tag{9}$$

The Riesz–Fréchet representation theorem for the dual of a Hilbert space furnishes a linear and isometric bijection $H : E' \to E$ having the property that, for all $x' \in E'$ and all $y \in E$, one has

$$x'(y) = (y \mid x) = (x \mid y),$$

where $x = H(x')$. So

$$x'(y)=(y\mid x)=(x\mid y). \tag{10}$$

Now, in (9) we define $\varepsilon : E \to E$, via $\varepsilon = H \circ \varepsilon_1$. The function ε is continuous and $\varepsilon(a)=0$.

So, for any u, v in E, one has (see (10))

$$\varepsilon_1(u)(v)=\left(H\left(\varepsilon_1(u)\mid v\right)\right)=\left(\varepsilon(u)\mid v\right) \tag{11}$$

In view of (9), (8) becomes (see (11)):

$$f(v)-f(u)=\left(df(a)+d^2f(a)(u-a)+\|u-a\|\varepsilon_1(u)\right)(v-u)+\|v-u\|\Omega(v)=$$
$$=df(a)(v-u)+d^2f(a)(u-a)(v-u)+\|u-a\|\varepsilon_1(u)(v-u)+\|v-u\|\Omega(v)$$
$$=df(a)(v-u)+d^2f(a)(u-a,v-u)+\|u-a\|\left(\varepsilon(u)\mid v-u\right)+\|v-u\|\Omega(v).$$

From the last equality and (7), we obtain

$$\|v-a\|\omega(v)-\|u-a\|\omega(u)=$$
$$d^2f(a)(u-a,v-u)+\|u-a\|\left(\varepsilon(u)\mid v-u\right)+\|v-u\|\Omega(v),$$

Because of (**), we write the last equality as follows:

$$\begin{aligned}&T(u-a,v-u)+\|u-a\|\omega(v)-\|v-a\|\omega(v)+\\&+\|u-a\|\left(\varepsilon(u)\mid v-u\right)=-\|v-u\|\Omega(v).\end{aligned} \tag{12}$$

Let us denote

$u-a=x$ and $v-u=t$.

Hence

$v-a=v-u+u-a=t+x,\ u=x+a,\ v=u+t=x+a+t$

and (12) becomes

$$T(x,t)+\|x\|\omega(x+a)-\|x+t\|\omega(x+a+t)+\|x\|\left(\omega(x+a)\mid t\right)=$$
$$-\|t\|\Omega(x+a+t)$$

Consequently, for any $0\neq t\in E$ and any $x\in E$:

$$\begin{aligned}&\frac{1}{\|t\|}\left(T(x,t)+\|x\|\omega(a+x)-\|x+t\|\omega(x+a+t)+\|x\|\left(\varepsilon(a+x)\mid t\right)\right)=\\&=-\Omega(a+x+t)\end{aligned} \tag{13}$$

We have

$$\lim_{v\to u}\Omega(v)=\lim_{v-u\to 0}\Omega(v)=0,$$

which means

$$\lim_{t\to 0}\Omega(a+x+t)=0$$

and (13) says that the proof is finished.

It is known that a bilinear, symmetric and continuous functional $T : E\times E\to\mathbb{R}$ generates the self adjoint operator $A: E\to E$ characterized by the fact that, for any x, y in E

$$T(x,y)=(A(x)\mid y)=(x\mid A(y))$$

and all self adjoint operators $A: E\to E$ are generated in this way.

Hence, we obtain the following alternative form of the previous result:

Proposition 1'. Let E be a real Hilbert space and $a\in E$. Let $A: E\to E$ be a self adjoint operator.

Then, there exist two continuous functions $\omega: E\to\mathbb{R}$, $\varepsilon: E\to E$, such that $\omega(a)=0,\ \varepsilon(a)=0$ and having the property that, for all $x\in E$, one has

$$\lim_{t\to 0}\frac{1}{\|t\|}\Big((A(x)\mid t)+\|x\|\omega(a+x)-\|x+t\|\omega(x+a+t)+\|x\|(\varepsilon(a+x)\mid t)\Big)=0$$

Considering Proposition 1 for the particular case when

$$T(x,y)=(x\mid y)$$

for all x, y in E, we obtain:

Proposition 2. Let E be a real Hilbert space and $a\in E$. Then, there exist two continuous functions $\omega: E\to\mathbb{R}$, $\varepsilon: E\to E$, such that $\omega(a)=0,\ \varepsilon(a)=0$ and having the property that, for all $x\in E$, one has

$$\lim_{t\to 0}\frac{1}{\|t\|}\Big((x\mid t)+\|x\|\omega(a+x)-\|x+t\|\omega(x+a+t)+\|x\|(\varepsilon(a+x)\mid t)\Big)=0$$

The reader is invited to consider Proposition 1 (respectively, Proposition 1') in the particular case when E = $\mathbb{R}^n$ with the usual scalar product

$$\big(x=(x_1,x_2,\dots,x_n)\mid y=(y_1,y_2,\dots,y_n)\big)=\sum_{i=1}^{n}x_iy_i$$

for $T:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ given via

$$T\big(x=(x_1,x_2,\dots,x_n)\mid y=(y_1,y_2,\dots,y_n)\big)=\sum_{i,j=1}^{n}a_{i,j}x_iy_j$$

(respectively for $A:\mathbb{R}^n\to\mathbb{R}^n$ given via

$$A\big(x=(x_1,x_2,\dots,x_n)\big)=y=(y_1,y_2,\dots,y_n),$$

where y_i = $y_i=\sum_{j=1}^{n}a_{ij}x_j$ for all $i=1,2,\dots,n$).

Here, the real matrix $(a_{ij})_{1\le i\le n,1\le j\le n}$ is symmetric, i.e. $a_{ij}=a_{ji}$ for all i and j.

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