RELATING THE RIEMANN HYPOTHESIS AND THE PRIMES BETWEEN TWO CUBES

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Abstract

In this paper we make an evaluation for the number of primes between two consecutive cubes, if we assume the Riemann hypothesis. There exists at least a prime between two consecutive cubes. More precisely, if we denote by N(n) the number of primes between $(n-1)^3$ *and* n^3*, then* $N(n) \approx n^2 / \log n$*.*

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1. **Intoduction**

In 1852 J. Bertrand postulated that for every integer $n \geq 2$ there is a prime between n and $2n$. Bertrand could not prove his postulate, but he verified it for all $n < 3000000$. Chebyshev was the first to prove the conjecture in 1852. This result was improved afterwards. For example, in 1989, N. Costa Pereira proved that if $x \geq 485492$, the interval $\left[x, \frac{258}{257} x \right)$ contains a prime.

The problem we study here comes from the intent to prove that between two consecutive squares there is at least a prime. This conjecture has not been neither proved nor infirmated until now. L. Skula made the hypothesis that for $n > 1$, each of the sequences $n^2 + 1, n^2 + 2, \ldots, n^2 + n$ and $n^2 + n + 1, n^2 + n + 2, \ldots, n^2 + 2n$ contains at least a prime.

A. Schinzel made an even stronger assumption, stating that for every real number $x \geq 8$ between x and $x + (\log x)^2$ there is at least a prime.

Sierpinski observed that from a result of A. E. Ingham it results that for m fixed and n large enough (depending of m), between n^3 and $(n+1)^3$ there exists at least m primes, that represents obviously a weaker result than the one presented here.

In this paper we evaluate the number of primes between $(n-1)^3$ and n^3.

We give now a list of the notations used:

- $\pi(x)$ is the number of primes less than or equal to *x;*
- $f(x)=o\left(g(x)\right)$, if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$ (where $g(x)\neq 0$ for large values of *x*);
- $f(x)\sim g(x)$, if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$ (where $g(x)\neq 0$ for large values of *x*);
- $\Gamma(s)=\int\limits_{0+0}^{\infty} x^{s-1}e^{-x}dx$, where $s>0$.

2. **The relation between the Riemann ζ function and the distribution of primes**

The Riemann ζ function is defined as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

where *s* is the complex variable $s=\sigma+it$, $\sigma,t\in\mathbb{R}$, $\sigma>1$.

The ζ function is analytic and has an analytic continuation for $\sigma>0$, excepting the point $s=1$, given by the formula

$$\zeta(s)=\frac{s}{s-1}-s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}dx.$$

In $s=1$ the ζ function has a simple pole with the residue 1.

When $\sigma<0$ the analytic continuation of ζ is given by the formula

$$\zeta(s)=2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s).$$

It results that all non-positive even numbers are roots for ζ, roots that are called trivial zeros.

It was proved that the non-trivial zeros of ζ are in the so called „critic strip” $0<\mathrm{Re}(s)<1$.

Riemann made the hypothesis that all the zeros of ζ are on the line $\mathrm{Re}(s)=\frac{1}{2}$, fact that is known as the „Riemann Hypothesis” and that has not been proved until now.

On the set of the zeros, an order has been introduced, as follows: $s_1<s_2$ if $\mathrm{Re}(s_1)<\mathrm{Re}(s_2)$, and if the real parts are equal, then $s_1<s_2$ if $\mathrm{Im}(s_1)<\mathrm{Im}(s_2)$. Van der Lune has proved that the first 1.500.000.000 zeros of ζ have the real part $\frac{1}{2}$.

The prime number theorem states that $\pi(x)\sim\frac{x}{\log x}$. This results taking into account the fact that $\zeta(s)\neq 0$, if $\mathrm{Re}(s)\geq 1$ and $s\neq 1$.

As new zeros of ζ are computed, we obtain more precise information for the prime distribution function π by getting finer inequalities.

For instance, after the first 25,000 zeros of ζ have been computed, in 1938, J. B. Rosser has proved that for $x \ge 55$ we have

$$\frac{x}{\log x + 2} < \pi(x) < \frac{x}{\log x - 4}.$$

In 1962, after computing the first 3,502,500 zeros Rosser and Schoenfeld have proved that

$$\pi(x) < \frac{x}{\log x - \frac{3}{2}}, \text{ for } x > e^{\frac{3}{2}} \text{ and}$$

$$\pi(x) > \frac{x}{\log x - \frac{1}{2}}, \text{ for } x \ge 67 .$$

3. **The main result**

Assuming the Riemann hypothesis, L. Schoenfeld proves that

$$|\pi(x) - \mathrm{li}(x)| < \frac{1}{8\pi}\sqrt{x}\log x \text{ for } x \ge 2657 \quad (1)$$

where $\mathrm{li}(x) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \left(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{\infty} \frac{dt}{\log t} \right)$.

We denote $N(n)$ the number of primes between $(n-1)^3$ and n^3.

We prove the following:

Theorem. If n is a positive integer, then $N(n) \sim \frac{n^2}{\log n}$, assuming the Riemann Hypothesis.

Proof:

From (1) we have

$$\mathrm{li}(x) - \frac{1}{8\pi}\sqrt{x}\log x < \pi(x) < \mathrm{li}(x) + \frac{1}{8\pi}\sqrt{x}\log x, \text{ for } x \ge 2657.$$

It results that

$$\pi\left(n^3\right) - \pi\left((n-1)^3\right) > \mathrm{li}\left(n^3\right) - \frac{3}{8\pi} n\sqrt{n}\log n - \mathrm{li}\left((n-1)^3\right) - \frac{3}{8\pi}(n-1)\sqrt{n-1}\log(n-1) =$$

$$= \int_{(n-1)^3}^{n^3} \frac{dt}{\log t} - \frac{3}{8\pi} n\sqrt{n}\log n - \frac{3}{8\pi}(n-1)\sqrt{n-1}\log(n-1) >$$

$$> \int_{(n-1)^3}^{n^3} \frac{dt}{\log t} - \frac{3}{4\pi} n\sqrt{n} \log n > \frac{n^3-(n-1)^3}{3\log n} - \frac{3}{4\pi} n\sqrt{n}\log n =$$

$$= \frac{3n^2-3n+1}{3\log n} - \frac{3}{4\pi} n\sqrt{n}\log n = \frac{n^2}{\log n} - \frac{3n-1}{3\log n} - \frac{3}{4\pi} n\sqrt{n}\log n,$$

for $(n-1)^3 \geq 2657$, therefore $n \geq 15$.

Thus

$$N(n) > \frac{n^2}{\log n} - \frac{3n-1}{3\log n} - \frac{3}{4\pi} n\sqrt{n}\log n. \qquad (2)$$

We evaluate the last two terms in (2).
We have

$$\frac{3n-1}{3\log n} = o\left(\frac{n^2}{\log n}\right) \text{ and}$$

$$n\sqrt{n}\log n = o\left(\frac{n^2}{\log n}\right).$$

Therefore, it results that

$$N(n) > \frac{n^2}{\log n} + o\left(\frac{n^2}{\log n}\right). \qquad (3)$$

From (1) it also results that

$$\pi\left(n^3\right) - \pi\left((n-1)^3\right) < \mathrm{li}\left(n^3\right) + \frac{3}{8\pi} n\sqrt{n}\log n - \mathrm{li}\left((n-1)^3\right) + \frac{3}{8\pi}(n-1)\sqrt{n-1}\log(n-1) =$$

$$= \int_{(n-1)^3}^{n^3} \frac{dt}{\log t} + \frac{3}{8\pi} n\sqrt{n}\log n + \frac{3}{8\pi}(n-1)\sqrt{n-1}\log(n-1) <$$

$$< \int_{(n-1)^3}^{n^3} \frac{dt}{\log t} + \frac{3}{4\pi} n\sqrt{n}\log n < \frac{n^3-(n-1)^3}{3\log(n-1)} + \frac{3}{4\pi} n\sqrt{n}\log n.$$

So we obtained that

$$N(n) < \frac{n^2}{\log(n-1)} - \frac{3n-1}{3\log(n-1)} + \frac{3}{4\pi} n\sqrt{n}\log n. \qquad (4)$$

Evaluating the terms in (4), we get

$$\frac{n^2}{\log(n-1)} \sim \frac{n^2}{\log n},$$

$$\frac{3n-1}{3\log(n-1)} = o\left(\frac{n^2}{\log n}\right) \text{ şi}$$

$$n\sqrt{n}\log n = o\left(\frac{n^2}{\log n}\right).$$

Therefore, we have

$$N(n) < \frac{n^2}{\log n} + o\left(\frac{n^2}{\log n}\right). \tag{5}$$

From (3) and (5) it results that

$$N(n) \sim \frac{n^2}{\log n}. \quad \square$$

Consequence. Assuming the Riemann Hypothesis, there exists at least a prime between two consecutive cubes.

Proof: If we assume the Riemann hypothesis, we have the relation (2)

$$N(n) > \frac{n^2}{\log n} - \frac{3n-1}{3\log n} - \frac{3}{4\pi} n\sqrt{n}\log n.$$

In order to prove that between two cubes there is at least a prime, it is enough to prove that

$$\frac{n^2}{\log n} - \frac{3n-1}{\log n} - \frac{3}{4\pi} n\sqrt{n}\log n > 0. \tag{6}$$

It is easy to see that for $n \geq 31$, we have

$$\frac{n^2}{10\log n} > \frac{3n+1}{3\log n}. \tag{7}$$

Therefore, it is enough to prove that

$$\frac{9n^2}{10\log n} > \frac{3}{4\pi} n\sqrt{n}\log n. \tag{8}$$

The previous relation is equivalent to

$$\sqrt[4]{n} > \sqrt{\frac{5}{6\pi}}\log n. \tag{9}$$

Denote $x = \sqrt[4]{n}$ and $k = 4\sqrt{\frac{5}{6\pi}} \simeq 2.0601$. The relation (9) is equivalent to $x > k\log x$.

Let $f:[1,\infty)\to\mathbb{R}$, $f(x)=x-k\log x$. We have $f'(x)=1-\frac{k}{x}>0$ for $x>k$, so *f* is increasing for $x>k$; as $f(k)>0$ it results that $f(x)>0$ for $x>k$ and, therefore, (8) is true for $n>k^4\simeq 18.011$.

Direct verifications for $1\le n\le 31$ lead us to state that between two consecutive cubes there exists at least a prime. □

4. **Remarks**

Even assuming the Riemann hypothesis, from Schoenfeld's inequality (1) it does not result that between $(n-1)^2$ and n^2 there exists at least a prime because we obtain that

$$\pi(n^2)-\pi((n-1)^2)>\frac{2n-1}{2\log n}-\frac{1}{2\pi}n\log n,$$

and the lower bound is non-positive for *n* large enough.

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