Moduli Spaces of Rank 3-stable Vector Bundles on Hirzebruch Surfaces with $c_1 = \sigma$[1]

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Abstract

We study the nonemptiness, the irreducibility, the smothness and the unirationality of the moduli spaces of rank 3 stable vector bundles on a Hirzebruch surface with $c_1 = \sigma$.
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1. *Introduction*

In [1] Li and Qin show that on a rational ruled surface X, the moduli space of vector bundles stable with respect to some suitable ample divisor is irreducible and unirational whenever the moduli space is nonempty. Moreover, it is proved in [1] that every irreducible component of an arbitrary moduli space is unirational. Based on this facts, in [2] is studied whether an arbitrary moduli space of rank-3 stable bundles is irreducible.

The ideea was the following one:

We consider a suitable ample divisor L_0 on X such that the moduli space of L_0-stable bundles is irreducible and unirational whenever it is nonempty; then, for an arbitrary ample divisor L on X, we take the moduli space of rank-3 bundles stable with respect to L. We compare the two moduli spaces of rank-3 bundles stable with respect to L_0 and L respectively by estimating the number of moduli of rank-3 vector bundles in their difference. From this estimate, we can conclude about the irreducibility and unirationality of the moduli space of rank-3 bundles stable with respect to L.

In [2] is analized the case of the moduli spaces of rank-3 vector bundles with the first Chern class $c_1 = 0$ and $c_1 = f + \sigma$. In this paper, following the approach from [2], we discuss the cases $c_1 = \sigma$.

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2. *Stability conditions for rank-3 vector bundles*

Let X be a rational ruled surface with the ruling $\pi : X \longrightarrow \mathbb{P}^1$, f a fiber of π, and σ a section to π such that $\sigma^2 = -e$. We denote by $K_X = -2\sigma - (2+e)f$ the canonical divisor of X. If $L = x\sigma + yf$ is an ample divisor, we note $r_L = \frac{y}{x}$.

Let c_1 be a divisor on X and c_2 an integer such that:

$$3c_2 - c_1^2 > 0,$$

and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two adjacent chambers of type $(3, c_1, c_2)$.

Let be V a rank-3 vector bundle L_1 stable (with $L_1 \in \mathcal{C}_1$), but L_2 unstable ($L_2 \in \mathcal{C}_2$) and the Harder-Narasimhan filtration of V with respect of L_2 given by:

$$0 \subset V_1 \subset ... \subset V_k \subset V.$$

From [2], we see that one of the next cases must happend:

1. $k = 1, F = c_1(V_1)$ and $i = rank(V_1) = 1$;
2. $k = 1, F = c_1(V_1)$ and $i = rank(V_1) = 2$;
3. $k = 2, F = c_1(V_2)$ and $i = rank(V_2) = 2$;
4. $k = 2, F = c_1(V_1)$ and $i = rank(V_1) = 1$.

Moreover, from [2] we know that for every rank-3 vector bundle L_1-stable and L_2-unstable, there exists an integer $0 < i < 3$ and a divisor F such that:

1. $(3F - ic_1) \cdot L_1 < 0 < (3F - ic_1) \cdot L_2$;
2. $-4 \cdot (3c_2 - c_1^2) \leq (3F - ic_1)^2 < 0$

By [2] we have the next result:

Theorem 2.1(see [2]) *Let* $\xi = (3F - c_1)$ *a nonempty wall of type* $(3; c_1, c_2)$ *and let* $\mathcal{C}$ *be a chamber of type* $(3; c_1, c_2)$ *such that it's closure* $\bar{\mathcal{C}}$ *intersects with* W^ξ *and that* $L_1 \cdot \xi < 0$ *for* $L_1 \in \mathcal{C}$. *Let* L_0 *be an ample divisor contained in* $(\bar{\mathcal{C}} \cap W^\xi)$. *Assume that* V *is a rank-3 bundle given by the nontrivial extension*

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow V \longrightarrow V' \longrightarrow 0,$$

where V' *is* L_0*-semistable such that no* L_0*-destabilizing rank-1 subsheaf can be lifted to a subsheaf of* V *and* V *has the Chern class* $c_1(V) = c_1$ *and* $c_2(V) = c_2$. *If* $c_1 \neq 0$ *[mod 3] and if* V' *is* L_1*-semistable, then* V *is* L_1*-stable.*

3. *The first case*

Let $V_1 = \mathcal{O}_X(F)$ and let $W = V/V_1$. By Proposition 1.2-[2], W is torsion free and L_2 semistable. We shall estimate the number of moduli of those V's coming from the next exact sequences:

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow V \longrightarrow W \longrightarrow 0 \qquad (1)$$

and

$$0 \longrightarrow W \longrightarrow W^{**} \longrightarrow Q \longrightarrow 0, \tag{2}$$

where Q is a sheaf supported on some 0-cycles in X. We have the next result from [2]:

Proposition 3.1(see [2]) *Fix a divisor F. Let $\xi = 3F - c_1$, and let*

$$d_\xi(c_1, c_2) = -\frac{3c_2 - c_1^2}{3} + 2 + \frac{\xi^2}{6} + \frac{K_X \cdot \xi}{2}.$$

Then, #(moduli of V)$\leq (6c_2 - 2c_1^2 - 8) + d_\xi(c_1, c_2)$.

Lemma 3.2 *Let $\xi = (3F - c_1)$.*
If $c_1 = \sigma$ then $d_\xi(c_1, c_2) \leq 0$. Moreover, $d_\xi(c_1, c_2) = 0$ if and only if $\xi = 2\sigma - 3c_2 f$.

Proof. Let $\xi = x\sigma - yf$. The inequality $\xi L_1 < 0 < \xi L_2$ implies that x and y have the same sign and none of them are zero. We are now able to evaluate the expression of

$$d_\xi(c_1, c_2) = -\frac{3c_2 - c_1^2}{3} + 2 + \frac{-x(ex - 3e + 6) - 2y(x - 3)}{6}.$$

Since $c_1 = \sigma$, $F = \frac{x+1}{3}\sigma - \frac{y}{3}f$ and we obtain $x \equiv 2[3]$ and $y \equiv 0[3]$. Let suppose that $x > 0$. This implies that $x \geq 2$ and $y > 0$. For $x \geq 5$, it's obvious that $d_\xi(c_1, c_2) < 0$.
If $x = 2$, then $d_\xi(c_1, c_2) = \frac{(e+y)-(3c_2-c_1^2)}{3}$.
Since $-4(3c_2 - c_1^2) \leq (3F - ic_1)^2 = \xi^2 = (x\sigma - yf)^2 = -4(e + y)$, we have $d_\xi(c_1, c_2) \leq 0$ with equality if and only if $x = 2$ and $y = 3c_2$, and so $\xi = 2\sigma - 3c_2 f$. If $x < 0$, then $x \leq -1$ and $y \leq -3$.
If $e \geq 1$, then $d_\xi(c_1, c_2) < 0$.
If $e = 0$, then

$$d_\xi(c_1, c_2) = -\frac{3c_2 - c_1^2}{3} + 2 + \frac{-6x - 2y(x - 3)}{6}.$$

Since $y \leq -3$, we always have

$$\frac{-6x - 2y(x - 3)}{6} \leq \frac{-6x + 6(x - 3)}{6} < -3,$$

and so

$$d_\xi(c_1, c_2) \leq -1.$$

Remark 3.3 Let $c_1 = \sigma$ and $\xi = 2\sigma - 3c_2 f$. Since $\xi = 3F - c_1 \Longrightarrow$

$$F = \sigma - c_2 f.$$

Since from the first exact sequence (1) we have the relations

$$c_1(V) = \mathcal{O}_X(F) + c_1(W) \text{ and } c_2 = Fc_1(W) + c_2(W),$$

it follows $c_1(W) = c_2 f$ and $c_2(W) = 0$.
By the second exact sequence (2) we get $c_2(W^{**}) + l(Q) = c_2(W) = 0$. Since W^{**} is L_2 semistable it results $c_2(W^{**}) \geq 0$, $c_2(W^{**}) = l(Q) = 0$ and W locally free. By the results in [4], W cannot be stable since $t = \frac{[4c_2(W) - c_1(W)^2]}{4} = 0$. In conclusion, W is L_2-semistable.
As in the proof of Lemma 2.2 from [2], W sit in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(\frac{c_2 f}{2}) \longrightarrow W \longrightarrow \mathcal{O}_X(\frac{c_2 f}{2}) \longrightarrow 0.$$

Since $Ext^1(\mathcal{O}_X(\frac{c_2 f}{2}), \mathcal{O}_X(\frac{c_2 f}{2})) = 0$ we obtain $W = W^{**} = \mathcal{O}_X(\frac{c_2 f}{2}) \oplus \mathcal{O}_X(\frac{c_2 f}{2})$. Thus, we can conclude that V sits in an exact sequence of type:

$$0 \longrightarrow \mathcal{O}_X(\sigma - c_2 f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2 f}{2}) \oplus \mathcal{O}_X(\frac{c_2 f}{2}) \longrightarrow 0$$

Conversely, if V is the bundle corresponding to an extension $e = (e_1, e_2)$ in

$$Ext^1(\mathcal{O}_X(\frac{c_2 f}{2})^2, \mathcal{O}_X(\sigma - c_2 f)) \cong H^1(X, \mathcal{O}(\sigma - \frac{3c_2 f}{2}))^{\oplus 2},$$

and according to Proposition 1.6 from [2] if both $e_i \neq 0$, then V is L'-stable where L' is contained in the chamber whose upper wall is W^ξ and all this V's are parametrized by a smooth unirational variety of dimension $(6c_2 - 2c_1^2 - 8)$.

4. ***The second case***

In this section we estimate the number of moduli of those rank-3 vector bundles V's with $c_1 = \sigma$ which satisfy case 2). We remind now the hypothesis of section 2.3 from [2]:

Let be V a rank-3 vector bundle given by the exact sequence:

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow \mathcal{O}_X(c_1 - F) \otimes I_Z, \tag{3}$$

where V_1 is a L_2 semistable rank-2 bundle. Dualizing the above sequence, we have:

$$0 \longrightarrow \mathcal{O}_X(F - c_1) \longrightarrow V^* \longrightarrow W \longrightarrow 0 \tag{4}$$

where W is torsion free and $W^* = V_1$. Since V^* is L_1 stable but L_2 unstable, the Harder-Narasimhan filtration of V^* with respect of L_2 is

$$0 \subset \mathcal{O}_X(F - c_1) \subset V^*.$$

In this case, $c_1(V^*) = -c_1$ and $c_2(V^*) = c_2$. V^* is now in the same hypothesis as the above section, so applying Proposition 3.1 we get:

$$\xi = 3(F - c_1) - (-c_1) = 3F - 2c_1$$

and

$$\#(\text{moduli of } V) = \#(\text{moduli of } V^*) \leq (6c_2 - c_1^2 - 8) + d_\xi(-c_1, c_2).$$

Lemma 4.1 *Let* $\xi = 3F - 2c_1$*. If* $c_1 = \sigma$*, then* $d_\xi(-c_1, c_2) \leq 0$*. Moreover* $d_\xi(-c_1, c_2) = 0$ *if and only if* $\xi = \sigma - \frac{3c_2-3}{2} f$*.*

Proof. Let $\xi = 3F - 2c_1 = x\sigma - yf$. From the same argument as in the above lemma x and y have the same sign and cannot be zero. By Proposition 3.1,

$$d_\xi(-c_1, c_2) = -c_2 + \frac{6 - 3x - e}{3} + \frac{(xe + 2y)(3 - x)}{6}.$$

If $c_1 = \sigma$, then

$$F = \frac{x+2}{3}\sigma - \frac{y}{3} f, \text{ and } x \equiv 1[3],\ y \equiv 0[3].$$

If $x \geq 3$, then $d_\xi(-c_1, c_2) \leq 0$. If $x < 3$, then $x = 1$ or $x < -2$.
Let $x = 1$. Then $d_\xi(-c_1, c_2) = -c_2 + \frac{2y+3}{3}$. Recall that $c_1(V_1) = F = \sigma - \frac{y}{3} f$ and from the exact sequence that gave us V we have the relation

$$c_2(V_1) \leq c_2 - F(c_1 - F) = c_2 - \frac{y}{3}.$$

By Lemma 1.10 in [4], since V_1 is L_2 stable, we have:

$$r_{L_2} \leq e + 2c_2(V_1) + e + \frac{y}{3} \leq 2(c_2 - \frac{y}{3}) + e + \frac{y}{3} = 2c_2 + e - \frac{y}{3}.$$

Since $\xi \cdot L_2 > 0$, it results that $r_{L_2} > e + y$. From the inequality

$$e + y < r_{L_2} < e + 2c_2 - \frac{y}{3}$$

we obtain that

$$\frac{2y}{3} < c_2,$$

and since $y \equiv 0[3]$ and $y > 0$, we get $d_\xi(-c_1, c_2) = -c_2 + \frac{2y}{3} + 1 < 1$ unless possibly $y = \frac{3c_2-3}{2}$, and thus

$$\xi = \sigma - \frac{3c_2 - 3}{2} f.$$

If $x \leq 0$, then $x \leq -2$ and $y \leq -3$. Since

$$d_\xi(-c_1, c_2) = -\frac{3c_2 - c_1^2}{3} + (2 + y) + \frac{x(3e - 6 - ex - 2y)}{6},$$

we get $d_\xi(-c_1, c_2) < 0$ if $e > 1$ and $d_\xi(-c_1, c_2) \leq -1$ if $e = 0$.

Remark 4.2 Let $c_1 = \sigma$ and $\xi = \sigma - \frac{3(c_1-1)}{2} f$. Since $\xi = 3F - 2c_1$, it results that

$$F = c_1(V_1) = \sigma + \frac{1 - c_2}{2} f.$$

From the above proof, we obtain

$$e + \frac{3(c_2 - 1)}{2} = e + y < r_{L_2} \leq e + 2c_2(V_1) + \frac{y}{3},$$

$$\text{and so } \frac{y}{3} < c_2(V_1).$$

$$\text{Thus } \frac{c_2 - 1}{2} < c_2(V_1) \Longleftrightarrow \frac{c_2 - 1}{2} + 1 \leq c_2(V_1) \Longleftrightarrow \frac{c_2 + 1}{2} \leq c_2(V_1).$$

From the first exact sequence in the set up it follows that

$$c_2(V_1) + l(Z) = c_2 - F(c_1 - F) \Longrightarrow$$

$$c_2(V_1) + l(Z) = \frac{c_2 + 1}{2}.$$

Therefore, from both relations we obtain $c_2(V_1) = \frac{c_2+1}{2}$ and $l(Z) = 0$, this implies $Z = \emptyset$. Thus V sits in the exact sequence:

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow \mathcal{O}_X(c_1 - F) \longrightarrow 0,$$

and so,

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2 - 1}{2}) \longrightarrow 0 \tag{5}$$

Remember that V_1 is L_2 semistable and

$$e + \frac{3(c_2 - 1)}{2} < r_{L_2} \leq e + \frac{2(c_2 + 1)}{2} + \frac{3(c_2 - 1)}{2} \Longleftrightarrow$$

$$e + \frac{3(c_2 - 1)}{2} < r_{L_2} \leq e + \frac{5c_2 - 1}{2}.$$

As in the proof of Proposition 3.4 in [4] and Remark 2.12 in [2], we verify that V_1 sits in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(\sigma - c_2 f) \longrightarrow V_1 \longrightarrow \mathcal{O}_X(\frac{1 + c_2}{2} f) \Longrightarrow 0 \tag{6}$$

According to [4], V_1 sits in an exact sequence of the type:

$$0 \longrightarrow \mathcal{O}_X(G) \longrightarrow V_1 \longrightarrow \mathcal{O}_X(c_1(V_1) - G) \Longrightarrow 0, \qquad (7)$$

where

$$2G = c_1(V_1) + \zeta \text{ and } \zeta = 2c_1(V_1) - 2c_2(V_1)f. \ (\bigstar).$$

Doing the substitutions in ($\bigstar$), it follows

$$\zeta = \sigma - \frac{3c_2 + 1}{2} \text{ and } G = \sigma - c_2 f,$$

and so the conclusion.

Now, we verify that V sits in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(\sigma - c_2 f) \longrightarrow V_1 \longrightarrow \mathcal{O}_X(\frac{c_2 - 1}{2} f) \oplus \mathcal{O}_X(\frac{c_2 + 1}{2} f) \longrightarrow 0 \qquad (8)$$

Since

$$V_1/\mathcal{O}_X(\sigma - c_2 f) = \mathcal{O}_X(\frac{1 + c_2}{2})f)$$

and

$$V/V_1 = \mathcal{O}_X(\frac{c_2 - 1}{2} f)$$

we have the next sequence:

$$0 \longrightarrow \mathcal{O}_X(\frac{1 + c_2}{2} f) \longrightarrow V/\mathcal{O}_X(\sigma - c_2 f) \longrightarrow \mathcal{O}_X(\frac{c_2 - 1}{2} f) \longrightarrow 0.$$

Note that since $Ext^1(\mathcal{O}_X(\frac{c_2-1}{2} f), \mathcal{O}_X(\frac{c_2+1}{2} f)) = 0$, the above extension splits. Therefore,

$$V/\mathcal{O}_X(\sigma - c_2 f) = \mathcal{O}_X(\frac{c_2 - 1}{2} f) \oplus \mathcal{O}_X(\frac{c_2 + 1}{2} f),$$

and our conclusion follows.

Proposition 4.3 *Let c_2 be odd. If $c_1 = \sigma$, then for generic bundles V_1 in (6), generic extensions V in (5) are L' stable, where L' is any ample divisor contained in the chamber whose upper wall is $W^{(\sigma - \frac{3(c_2-1)}{2} f)}$. Moreover, all such V's are parametrized by a smooth rational variety of dimension $(6c_2 - 2c_1^2 - 8)$.*

Proof. If $\xi = \sigma - \frac{3(c_2-1)}{2}$, then by Theorem B in [4], for c_2 fixed, all nonempty moduli spaces $\mathcal{M}_L(2, \sigma + \frac{1-c_2}{2} f, \frac{c_2+1}{2})$ are birational, or in our case $V_1 \in \mathcal{M}_L(2, \sigma + \frac{1-c_2}{2} f, \frac{c_2+1}{2})$. Moreover, the Proposition 3.4, in [4] implies that the generic bundles V_1 in (6), are both L' stable and L_0 stable, where $L_0 \in W^{(\sigma - \frac{3(c_2-1)}{2} f)}$. Dualizing (5), we see that for bundles V_1, which are stable with respect to both L' and L_0 nontrivial extension V in (5), are L' stable. Moreover, for a fixed L_0- stable bundles V_1, we see that

$$Hom(V_1, \mathcal{O}_X(\frac{c_2-1}{2}f)) = H^0(X, V_1^* \otimes \mathcal{O}_X(\frac{c_2-1}{2}f)) = 0,$$

$\dim Hom(V_1, V) = 1$, and all the trivial extensions V in (6), are parametrized by $\mathbb{P}(Ext^1(\mathcal{O}_X(\frac{c_2-1}{2}f), V_1))$. Since all the L_0 stable V_1's in (5) are parametrized by an open dense subset in

$$\mathbb{P}(Ext^1(\mathcal{O}_X(\frac{c_2+1}{2}f), \mathcal{O}_X(\sigma - c_2 f)),$$

all such V's are parametrized by a smooth rational variety.

5. *The third case*

We recall the set up of section 2.4. in [2]. Let $c_1(V_1) = F_1$ and $c_1(V_2) = F_2 = F$. Therefore, we have the exact sequences:

$$0 \longrightarrow V_2 \longrightarrow V \longrightarrow \mathcal{O}_X(c_1 - F_2) \otimes I_{Z_2} \longrightarrow 0 \tag{9}$$

$$0 \longrightarrow \mathcal{O}_X(F_1) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(F_2 - F_1) \otimes I_{Z_1} \longrightarrow 0. \tag{10}$$

From the proof of Lemma 1.4 in [2], we have

$$(2F_1 - F_2) \cdot L_1 < 0 < (2F_1 - F_2) \cdot L_2,$$

and so from the second exact sequence

$$c_1(V_2)^2 - 4c_2(V_2) \leq (2F_1 - F_2)^2 < 0.$$

To start with, we remind the Proposition 2.20 from [2].

Proposition 5.1(see [2]) *Fix* $F = F_2 = c_1(V_2)$. *Let* $\eta = (3F - 2c_1)$, *and let*

$$d^*_\eta(c_1, c_2) = -\frac{2(c_2 - c_1^2)}{3} + 3 + \frac{(2F_1 - F_2)^2}{4} + \frac{(2F_1 - F_2) \cdot K_X}{2} + \frac{\eta^2}{12} + \frac{\eta \cdot K_X}{2}.$$

Then,

$$\#(\text{moduli of } V) \leq (6c_2 - 2c_1^2 - 8) + d^*_\eta(c_1, c_2).$$

Lemma 5.2 *Put* $\eta = (3F - 2c_1)$ *If* $c_1 = \sigma$, *then* $d^*_\eta(c_1, c_2) < 0$, *unless possibly either of the following:*

$$\eta = \sigma - \frac{3c_2}{2} f \ \text{ or } \ \eta = \sigma - \frac{3c_2 - 3}{2} f.$$

Proof. Put $(2F_1 - F_2) = x_1\sigma - y_1 f$ and $\eta = x_2\sigma - y_2 f$. From the same argument as in the proof of Lemma 2.21 in [2], x_i and y_i must have the same sign and neither of them can be zero. By the above proposition,

$$d^*_\eta(c_1, c_2) = -\frac{2(c_2 - c_1^2)}{3} + 3 + d_1 + d_2,$$

where

$$d_1 = \frac{(2F_1 - F_2)^2}{4} + \frac{(2F_1 - F_2) \cdot K_X}{2} = \frac{-x_1(ex_1 - 2e + 4) - 2y_1(x_1 - 2)}{4},$$

$$d_2 = \frac{\eta^2}{12} + \frac{\eta \cdot K_X}{2} = \frac{-x_2(ex_2 - 6e + 12) - 2y_2(x_2 - 6)}{12}.$$

Also, from the proof of the Lemma 2.21 in [2] we know that

$$d_1 - \frac{3c_2 - c_1^2}{3} \leq \frac{\eta^2}{12} - 1.$$

Since $\eta = (3F - 2c_1) = x_2\sigma - y_2 f$, it follows

$$F = \frac{x_2 + 2}{3}\sigma - \frac{y_2}{3}f, \;\; x_2 \equiv 1[3] \;\text{ and }\; y_2 \equiv 0[3].$$

Let's estimate d_2. If $x_2 > 0$, then $x_2 = 1$, or $x_2 = 4$ or $x_2 \geq 7$.
If $x_2 \geq 7$ then

$$d_2 \leq \frac{-7(e + 12) - 2y_2}{12} = \frac{-7e - 2y_2}{12} - 7 < -7,$$

and so

$$d^*_\eta(c_1, c_2) < 0.$$

If $x_2 = 4$, then

$$d_2 \leq \frac{-4(-2e + 12) + 4y_2}{12} = \frac{8e + 4y_2}{12} - 7 < -7,$$

since $-4(3c_2 - c_1^2) \leq \eta^2 = -16e - 8y_2 \Longrightarrow$

$$\frac{3c_2 - c_1^2}{3} \geq \frac{8e + 4y_2}{12}.$$

Since

$$d_2 = \frac{-4(-2e + 12) + 4y_2}{12} = \frac{8e + 4y_2}{12} - 7 = \frac{\eta^2}{12} - 7 < -7,$$

we get

$$d^*_\eta(c_1, c_2) < 0.$$

Let $x_2 = 1$; then $d_2 = \frac{(5e+10y_2)}{12} - 1$. According to the initial hypothesis, W^η

is the only wall of type $(3, c_1, c_2)$ separeting L_1 and L_2. Since V is L_1 stable, V is L_0 semistable by Theorem 1.6 and Remark 1.10 in [1], then

$$d_2 = \frac{5e + y_2}{12} - 1.$$

As in the proof of Theorem 3.1 in [1], we have

$$r_{L_0} < e + \frac{3(c_2 + 1)}{2},$$

and from the fact that $L_0\eta = 0$, it follows $r_{L_0} = e + y_2$; thus $y_2 < \frac{3c_2+3}{2}$. Since $y_2 \equiv 0[3] \Longrightarrow 2y_2 \equiv 0[3] \Longrightarrow 2y_2 < 3c_2 + 3 \Longrightarrow 2y_2 \leq 3c_2$.
Note that if $x_1 \neq 1$, then $d_1 \leq -2$, and so $d^*_\eta(c_1, c_2) < 0$. If $x_1 = 1$, then

$$d_1 - \frac{(3c_2 - c_1^2)}{3} \leq \frac{\eta^2}{12} - 1,$$

thus:

$$d^*_\eta(c_1, c_2) = -\frac{(c_2 - c_1^2)}{3} + 3 + (\frac{\eta^2}{12} - 1) + (\frac{(5e + 10y_2)}{12} - 1) \leq$$

$$\leq \frac{-12c_2 + 12 + 8y_2}{12} = \frac{-3c_2 + 3 + 2y_2}{3} \leq \frac{-3c_2 + 3 - 3c_2}{3} = 1.$$

Thus,

$$d^*_\eta(c_1, c_2) = 1 \Longleftrightarrow x_2 = 1 \text{ and } y_2 = 3c_2/2,$$

$$d^*_\eta(c_1, c_2) = 0 \Longleftrightarrow x_2 = 1 \text{ and } y_2 = (3c_2 - 3)/2.$$

For $x_2 < 0$, we always obtain $d^*_\eta(c_1, c_2) < 0$.

Remark 5.3 Let $c_1 = \sigma$ and $\eta = \sigma - \frac{3c_2}{2}f$. Since $\eta = 3F - 2c_1$, it follows

$$c_1(V_2) = F = \sigma - \frac{c_2}{2}f.$$

If $L_0 \in W^\eta$, then V is L_0 semistable; that implies

$$\frac{c_1(V_2)L_0}{2} = \frac{c_1L_0}{3},$$

and thus

$$r_{L_0} = e + \frac{3c_2}{2}.$$

By the Lemma 1.10 in [4], it follows:

$$e + \frac{3c_2}{2} = r_{L_0} \leq 2c_2(V_2) + e + \frac{c_2}{2}.$$

Therefore,
$$\frac{c_2}{2} \le c_2(V_2),$$
and since from the first exact sequence $c_2(V_2) + l(Z_2) = \frac{c_2}{2}$, it follows
$$c_2(V_2) = \frac{c_2}{2} \text{ and } Z_2 = \emptyset.$$
As in the remark, we verify that V_2 sits in the exact sequence
$$0 \longrightarrow \mathcal{O}_X(\sigma - c_2 f) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(\frac{c_2}{2} f),$$
and V sits in the exact sequence
$$0 \longrightarrow \mathcal{O}_X(\sigma - c_2 f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2}{2} f) \oplus \mathcal{O}_X(\frac{c_2}{2} f).$$

Remark 5.4 Let $c_1 = \sigma$ and $\eta = \sigma - \frac{3c_2-3}{2} f$. Since $\eta = 3F - 2c_1$, we obtain $c_1(V_2) = F = F_2\sigma - \frac{c_2-1}{2} f$. From the above proposition we have $d^*_\eta(c_1, c_2) \le 0$. If $d^*_\eta(c_1, c_2) = 0$, then
$$d_1 - \frac{(3c_2 - c^2 - 1)}{3} = \frac{\eta^2}{12} - 1 = \frac{-(e + 3c_2 - 3)}{12} - 1.$$
On the other hand,
$$d_1 = \frac{e + 2y_1}{4} - 1 \text{ since } x_1 = 1 \text{ in this case.}$$
So, using this equalities we obtain $y_1 = \frac{3c_2+1}{2}$.

Since $2F_1 - F_2 = x_1\sigma - y_1$, it results $F_1 = \sigma - c_2 f$. From the first exact sequence we have
$$c_2(V_2) = \frac{1 + c_2}{2} + l(Z_1),$$
and from the second exact sequence
$$c_2(V_2) + l(Z_2) = \frac{c_2 + 1}{2}$$
it follows
$$c_2(V_2) = \frac{c_2 + 1}{2} \text{ and } Z_1 = Z_2 = \emptyset.$$
We conclude that the two exact sequences in (9) and (10) are:
$$0 \longrightarrow V_2 \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2 - 1}{2} f) \longrightarrow 0.$$
$$0 \longrightarrow \mathcal{O}(\sigma - c_2 f) \longrightarrow V_2 \longrightarrow \mathcal{O}_X(\frac{c_2 + 1}{2} f) \longrightarrow 0.$$
So we have:
$$0 \longrightarrow \mathcal{O}(\sigma - c_2 f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2 - 1}{2} f) \oplus \mathcal{O}_X(\frac{c_2 + 1}{2} f) \longrightarrow 0.$$

6. *The fourth case*

We begin this section, reminding the hypothesis from [2]-2.5. Let $F_1 = c_1(V_1) = F$ and $F_2 = c_1(V_2)$. The rank-3 vector bundle satisfying case d is sitting in the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(F_1) \longrightarrow V \longrightarrow V/\mathcal{O}_X(F_1) \longrightarrow 0,$$

where $V/\mathcal{O}_X(F_1)$ is coming from the exact sequence:

$$0 \longrightarrow V_2/\mathcal{O}_X(F_1) \longrightarrow V/\mathcal{O}_X(F_1) \longrightarrow \mathcal{O}_X(c_1 - F_2) \otimes I_Z \longrightarrow 0,$$

with the remark from the proof of the Lemma 1.4 that:

$$(2F_2 - F_1 - c_1) \cdot L_1 < 0 < (2F_2 - F_1 - c_1) \cdot L_2.$$

Dualizing the second exact sequence it follows:

$$0 \longrightarrow [V/\mathcal{O}_X(F_1)]^* \longrightarrow V^* \longrightarrow \mathcal{O}_X(-F_1) \otimes I_{Z_2} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(F_2 - c_1) \longrightarrow [V/\mathcal{O}_X(F_1)]^* \longrightarrow \mathcal{O}_X(F_1 - F_2) \otimes I_{Z_1} \longrightarrow 0.$$

Remark that V^* is L_1-stable, but L_2 unstable and

$$2(F_2 - c_1) - c_1([V/\mathcal{O}_X(F_1)]^*) = (2F_2 - F_1 - c_1).$$

From the subsection d of [2] we know that: $\#(\text{moduli of } V) = \#(\text{moduli of } V^*) \leq (6c_2 - 2c_1^2 - 8) + d^*_\eta(-c_1, c_2)$, where $\eta = 3c_1([V/\mathcal{O}_X(F_1)]^*) - 2c_1(V^*) = (3F_1 - c_1)$ and
$d^*_\eta(-c_1, c_2) = -\frac{2(3c_2 - c_1^2)}{3} + 3 + \frac{(2F_2 - F_1 - c_1)^2}{4} + \frac{(2F_2 - F_1 - c_1) \cdot K_X}{2} + \frac{\eta^2}{12} + \frac{\eta \cdot K_X}{2}$.

Lemma 6.1 *Put $\eta = 3F_1 - c_1$. If $c_1 = \sigma$, then $d^*_\eta(-c_1, c_2) < 0$ unless $\eta = 2\sigma - 3c_2 f$.*

Proof. The proof is analogue as the proof of Lemma 5.2. Put $2F_2 - F_1 - c_1 = x_1\sigma - y_1 f$ and $\eta = 3F_1 - c_1 = x_2\sigma - y_2 f$, where x_i, y_i have the same sign and all of them are not 0 (from the same argument as in the previous lemmas).
If $c_1 = \sigma$, then

$$F_1 = \frac{x_2 + 1}{3}\sigma - \frac{y_2}{3} f,$$

and so

$$x_2 \equiv 2[3] \text{ and } y_2 \equiv 0[3].$$

Remember that $3c_1 - c_1^2 > 0$. Let

$$d_1 = \frac{(2F_2 - F_1 - c_1)^2}{4} + \frac{(2F_2 - F_1 - c_1) \cdot K_X}{2} = \frac{-x_1(ex_1 - 2e + 4) - 2y_1(x_1 - 2)}{4}$$

and

$$d_2 = \frac{\eta^2}{12} + \frac{\eta \cdot K_X}{2} = \frac{-x_2(ex_2 - 6e + 12) - 2y_2(x_2 - 6)}{4}.$$

From the proof of Lemma 2.21 in [2] we know that:

$$d_1 - \frac{3c_2 - c_1^2}{3} < \frac{\eta^2}{12} - 1.$$

Let's evaluate the expression of d_2.

If $x_2 > 2$, then $x_2 = 5$ or $x_2 \geq 8$ and $y_2 \geq 3$

If $x_2 \geq 8$, then

$$d_2 \leq \frac{-8(2e+12) - 4y_2}{12} = \frac{-4e}{3} - \frac{y_2}{3} - 8 < -8.$$

If $x_2 = 5$, then

$$d_2 = \frac{-5(-e+12) + 2y_2}{12} = \frac{5e + 2y_2}{12} - 5.$$

Since

$$-4(3c_2 - c_1^2) \leq \eta^2 = 25e - 10y_2 \iff$$

$$\frac{4(3c_2 - c_1^2)}{5} \geq 5e + 2y_2,$$

and so

$$d_2 < \frac{(3c_2 - c_1^2)}{15} - 5.$$

It follows that:

$$d_\eta^*(-c_1, c_2) \leq -\frac{(3c_2 - c_1^2)}{3} + 3 + (\frac{\eta^2}{12} - 1) + \frac{(3c_2 - c_1^2)}{12} - 5 \leq -3.$$

If $x_2 = 2$, then $d_2 = \frac{2e+2y_2}{3} - 2$. Since V is L_0 semistable, we get $L_0 \cdot \eta = 0$ and so $r_{L_0} = e + \frac{y_2}{2}$. As in the proof of Theorem 3.1 in [4], we obtain $r_{L_0} < e + \frac{3(c_2+1)}{2}$, and so

$$y_2 < 3(c_2 + 1).$$

Since $y_2 \equiv 0[3]$, it follows $y_2 \leq 3c_2$.

If $x_1 \neq 1$ then $d_1 \leq -2$ and so $d_\eta^*(-c_1, c_2) < 0$.

If $x_1 = 1$, then $d_1 - \frac{3c_2 - c_1^2}{3} \leq \frac{\eta^2}{12} - 1 = -\frac{e+y_2}{3} - 1$. Therefore:

$$d_\eta^*(-c_1, c_2) \leq -\frac{(3c_2 - c_1^2)}{3} + 3 + (-\frac{e + y_2}{3} - 1) + (\frac{2e + 2y_2}{3} - 2)$$

$$= -\frac{(3c_2 - c_1^2)}{3} + \frac{e + y_2}{3}$$

$$= \frac{-3c_2 - e + e + y_2}{3} \leq \frac{-3c_2 + 3c_2}{3} = 0.$$

Thus $d_\eta^*(-c_1, c_2) < 0$ unless $y_2 = 3c_2$. Let $x_2 < 0$ and $e \geq 1$.

If $x_2 \leq -7$, then

$$d_2 \leq \frac{7(-13e+12)+26y_2}{12} < -7,$$

and so $d^*_\eta(-c_1, c_2) < 0$.

If $x_2 = -4$, then

$$d_2 = \frac{4(-10e+12)+20y_2}{12} < -4,$$

and thus $d^*_\eta(-c_1, c_2) < 0$.

If $x_2 = -1$, then

$$d_2 = \frac{(-7e+12)+14y_2}{12} \leq -3,$$

and so $d^*_\eta(-c_1, c_2) < 0$.

If $e = 0$, then

$$d_2 = \frac{-12x_2 - 2y_2(x_2-6)}{12},$$

and we get $d^*_\eta(-c_1, c_2) \leq -1$.

Remark 6.2 Let $c_1 = \sigma$ and $\eta = 2\sigma - 3c_2 f$. Since $\eta = 3F_1 - c_1$, we get $F = \sigma - c_2 f$; thus $c_1(V/\mathcal{O}_X(F_1)) = c_1(V) - F_1 = c_2 f$, and $c_2(V/\mathcal{O}_X(F_1) = 0$. As in Remark 3.3, we get that $V/\mathcal{O}_X(F_1)$ sit in the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(\frac{c_2}{2}f) \longrightarrow V/\mathcal{O}_X(F_1) \longrightarrow \mathcal{O}_X(\frac{c_2}{2}f) \longrightarrow 0,$$

and V is given by:

$$0 \longrightarrow \mathcal{O}_X(\sigma - c_2 f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2}{2}f) \oplus \mathcal{O}_X(\frac{c_2}{2}f) \longrightarrow 0.$$

7. *Moduli spaces of rank 3 vector bundles with* $c_1 = \sigma$

The main results of this paper are given by the next theorems:

Theorem 7.1 *Let c_2 an even integer. If $c_1 = \sigma$ then:*

1. *For $c_2 < 2$, all the moduli spaces $\mathcal{M}_L(3, \sigma, c_2)$ are empty.*
2. *For $c_2 \geq 2$ and $r_L \geq e + \frac{3c_2}{2}$ or $r_L \leq \frac{2}{3c_2}$, the moduli space $\mathcal{M}_L(3, \sigma, c_2)$ is empty.*
3. *For $c_2 \geq 2$ and $\frac{2}{3c_2} < r_L < e + \frac{3c_2}{2}$, then the moduli space $\mathcal{M}_L(3, \sigma, c_2)$ is irreducible, smooth and unirational with dimension $(6c_2 + 2e - 8)$. Moreover, in this case a generic bundle in $\mathcal{M}_L(3, \sigma, c_2)$ sits in an exact sequence:*

$$0 \longrightarrow \mathcal{O}_X(\sigma - c_2 f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2 f}{2}) \oplus \mathcal{O}_X(\frac{c_2 f}{2}) \longrightarrow 0 \qquad (11)$$

Proof. The idea of the proof is the same as in the case of Theorem 3.2 in [2]. We remind that $\mathcal{M}_H(3,\sigma,c_2)$ is empty for $r_H >> 0$. If $\mathcal{M}_L(3,\sigma,c_2)$ is nonempty for some L, then all the bundles in $\mathcal{M}_L(3,\sigma,c_2)$ are not H stable. Therefore, the generic bundles in an irreducible component of $\mathcal{M}_L(3,\sigma,c_2)$ must come from some nonempty wall W^ξ as in the section 2. Since c_2 is even, then we have the next cases:

1. $\xi = 2\sigma - 3c_2 f$ or $\xi = \sigma - \frac{3c_2}{2} f$.
2. $e = 0$ and $\xi = -3c_2\sigma + 2f$.
3. If $c_2 < 2$ and c_2 even, it follows $c_2 \leq 0$ and so all ξ are effective.

This a contradiction since $\xi \cdot L_0 < 0$ for all ample divisors L_0.
If $e \geq 1$, since $r_L > e \geq 1$ and $c_2 \geq 2$, then we have always $r_L > \frac{2}{3c_2}$. If $r_L \geq e + \frac{3c_2}{2}$, then this is equivalent with $L \cdot (2\sigma - 3c_2 f) \geq 0$. As in the first part of the proof we get that $\mathcal{M}_L(3,\sigma,c_2)$ is empty.
If $r_L < e + \frac{3c_2}{2}$. We denote by L' a line bundle contained in the chamber whose upper wall is $W^{2\sigma - 3c_2 f}$.
From the main lemmas (3.2-section3, 4.1-section 4, 5.2-section 5, and 6.1-section 6) of the above sections, we know that if W^ξ separates L' and L, then the number of the moduli of bundles coming from W^ξ is less than $(6c_2 + 2e - 8)$. Therefore, $\mathcal{M}_{L'}(3,\sigma,c_2)$ can be identified with $\mathcal{M}_L(3,\sigma,c_2)$ just by removing some subschemes of codimension at least one. Remark 3.3 implies that $\mathcal{M}_{L'}(3,\sigma,c_2)$ is irreducible, and unirational, and a generic bundle in $\mathcal{M}_{L'}(3,\sigma,c_2)$ sits in (11). Thus for the conclusion for the moduli spaces $\mathcal{M}_L(3,\sigma,c_2)$ follows. If $e = 0$, then as in the above discussion $\mathcal{M}_L(3,\sigma,c_2)$ is empty if $r_L \geq \frac{3c_2}{2}$ or $r_L \leq \frac{2}{3c_2}$. If $\frac{2}{3c_2} < r_L < \frac{3c_2}{2}$, then again we can show the conclusion as we did in case $e \geq 1$.

The proof of the next result is a straightforward version of the above theorem.

Theorem 7.2 *Let c_2 an odd integer. If $c_1 = \sigma$ then:*

1. *For $c_2 < 3$, then all the moduli spaces $\mathcal{M}_L(3,\sigma,c_2)$ are empty.*
2. *For $c_2 \geq 3$ and $r_L \geq e + \frac{3c_2}{2}$ or $r_L \leq \frac{2}{3c_2}$, the moduli space $\mathcal{M}_L(3,\sigma,c_2)$ is empty.*
3. *For $c_2 \geq 3$ and $\frac{2}{3c_2} < r_L < e + \frac{3c_2}{2}$, then the moduli space $\mathcal{M}_L(3,\sigma,c_2)$ is irreducible, smooth and rational with dimension $(6c_2+2e-8)$. Moreover, in this case a generic bundle in $\mathcal{M}_L(3,\sigma,c_2)$ sits in an exact sequence:*

$$0 \longrightarrow \mathcal{O}_X(\sigma - c_2 f) \longrightarrow V \longrightarrow \mathcal{O}_X(\frac{c_2-1}{2} f) \oplus \mathcal{O}_X(\frac{c_2+1}{2} f) \longrightarrow 0.$$

References

1. W. Ping, Z. Qin, *Stable Vector Bundles on Algebraic Surfaces*, "Transactions of the American Mathematical Society." 345:833–852, 1994.
2. W.P. Li, Z. Qin, *Rank–3 Stable Bundles on Rational Ruled Surfaces*, "Mathematische Zeitschrift," 222:279-303, 1996.
3. Z. Qin, *Equivalence Classes of Polarizations and Moduli Spaces of Sheaves*, "Journal of Differential Geometry," 37:397-415, 1993.
4. Z. Qin, *Moduli Spaces of Stable Rank-2 Bundles on Ruled Surfaces*, "Invent. Math.", 110:615-626, 1992.