Attractors of topological iterated function system

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Abstract
The aim of this paper is to prove that the union of a Peano space and a segment with whose intersection is nonempty is the attractor of a topological iterated function system formed by three functions.
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1. Introduction

The iterated function systems were introduced by John Hutchinson in ([4]) and it turned out that their attractors had very interesting topological properties. The iterated function systems play an important role in the study of image compressing field. Also, the topological properties of the attractors such as connecteedness, arcwise connectedness or locally connectedness were studied by many mathematicians and important results were obtained ([1], [3], [5], [6], [7], [8], [9]).

In this paper we turn our interest towards the topological iterated function systems, which represent a generalization of the notion of iterated function system. The aim of this paper is to prove that a certain class of compact spaces can be attractors of some topological iterated function system. We consider any Peano space and then we take his union with a segment with whose intersection is nonempty. These sets form a class of compact spaces that can be attractors of topological iterated function systems formed by three function. The whole construction is described in the second part of the paper.

We consider a topological separated space (X,τ) and by $\mathcal{K}(X)$ we will understand the set of nonempty compact subsets of X. We have the following well known definitions:

Definition 1.1. Let (X,d) a metric space. The *Lipschitz constant* of a function $f : X \longrightarrow X$ is $Lip(f) = \sup\limits_{x,y\in X,\ x\neq y} \frac{d(f(x),f(y))}{d(x,y)}$. In the case when $Lip(f) < 1$, then f is called a contraction.

Definition 1.2. An *iterated function system* on (X,d) consists on a finite family of contraction $\{f_i\}_{i=\overline{1,n}}$ on X.

Definition 1.3. The *Hausdorff metric* is defined to be the following application $h:\mathcal{K}(X)\longrightarrow\mathcal{K}(X)$, $h(A,B)=\max\{d(A,B),d(B,A)\}$, where $d(A,B)=\sup\limits_{x\in A}\left(\inf\limits_{y\in B}d(x,y)\right)$.

We consider now the generalization of an iterated function system from a topological point of view, that is:

Definition 1.4. *([12])* A *topological iterated function system* consists on a finite family of continuous functions $\{f_j\}_{j=\overline{1,n}}$ defined on a topological separated space (X,τ) with the following properties:

a). For every compact subset $A\subset X$ there is a compact subset $B\subset X$ such that $A\subset B$ and $\overset{n}{\underset{j=1}{\cup}}f_j(B)\subset B$.

b). For every compact subset $A\subset X$ such that $\overset{n}{\underset{j=1}{\cup}}f_j(A)\subset A$, the intersection $\underset{n\geq 1}{\cap}f_{j_1j_2\ldots j_n}(A)$ is formed by one point for every $j_1,j_2,...,j_n\in\{1,2,...,n\}$.

Notation 1.1. a). A topological iterated function system will be denoted by $\mathcal{S}=(X,\{f_j\}_{j=\overline{1,n}})$.

b). For a topological iterated function system $\mathcal{S}=(X,\{f_j\}_{j=\overline{1,n}})$, the function $F_{\mathcal{S}}:\mathcal{K}(X)\longrightarrow\mathcal{K}(X)$ is defined by $F_{\mathcal{S}}(A)=\overset{n}{\underset{j=1}{\cup}}f_j(A)$.

c). The attractor of a topological iterated function system is a set $A\in\mathcal{K}(X)$ such that $F_{\mathcal{S}}(A)=A$.

The existence and uniqueness of the attractor of a topological iterated function system is assured by the following theorem.

Theorem 1.1. *([12]) Let (X,d) be a complete metric space and $\mathcal{S}=(X,\{f_k\}_{k=\overline{1,n}})$ a topological iterated function system. Then there exists a unique set $A\in\mathcal{K}(X)$ such that $F_{\mathcal{S}}(A)=A$.*

2. **Main results**

In this section we prove that the union of a Peano space and a segment with whose intersection is nonempty is the attractor of a topological iterated function system formed by three functions. For that we need to define first a Peano space.

Definition 2.1. A *Peano space* is a locally connected, connected and compact space.

In ([2]) it is given a well known characterization of Peano spaces, which is that Peano spaces are images of continuous functions.

Theorem 2.1. *([2]) Let K be a Peano space. Then there exists $f : [0,1] \longrightarrow K$ a continuous and surjective function.*

We consider a topological separated space (X, τ) and $t : [0,1] \longrightarrow X$ a continuous function. Then $K = t([0,1])$ is a Peano space. Let also be a function $f : [0,1] \longrightarrow X$ continuous and injective. We consider now the space $H = K \cup f([0,1])$ where $K \cap f([0,1]) = \{f(0)\}$ and, moreover, we suppose that $t(0) = f(0)$.

Theorem 2.2. *Let the space H be defined as above. Then H is the attractor of a topological iterated function system formed by three functions.*

Proof: It is well known that the interval $[0,1]$ is the attractor of the iterated function system $\mathcal{S} = \left([0,1], \left\{f_1(x) = \frac{x}{2},\ f_2(x) = \frac{x+1}{2}\right\}\right)$. Let $p : H \longrightarrow f([0,1])$ be defined by:

$$p(x) = \begin{cases} f(0), \text{ if } x \in K \\ x, \text{ if } x \in f([0,1]) \end{cases}$$

We define the following two functions $F_i : H \longrightarrow f([0,1]) \subset H$ for every $x \in H$ by $F_i(x) = (f \circ f_i \circ f^{-1} \circ p)(x)$ where $i \in \{1, 2, \}$. We obtain that $F_1(f([0,1])) \cup F_2(f([0,1])) = f([0,1])$ since:

$$F_1(f([0,1])) = f(f_1(f^{-1}(p(f([0,1]))))) = \\ f(f_1(f^{-1}(f([0,1])))) = f(f_1([0,1])) = f\left(\left[0, \tfrac{1}{2}\right]\right)$$

and

$$F_2(f([0,1])) = f(f_2(f^{-1}(p(f([0,1]))))) = \\ f(f_2(f^{-1}(f([0,1])))) = f(f_2([0,1])) = f\left(\left[\tfrac{1}{2}, 1\right]\right).$$

On the other hand, since K is a Peano space, there exists $t : [0,1] \longrightarrow K$ a continuous and surjective function as we already considered above and, moreover, we can suppose that $t(0) = f(0)$. We define now the third function $F_3 : H \longrightarrow H$ by:

$$F_3(x) = \begin{cases} f(0), \text{ if } x \in K \\ (t \circ f^{-1})(x), \text{ if } x \in f([0,1]) \end{cases}$$

We remark from the definitions of the functions F_1, F_2, F_3 that they satisfy $F_1(H) = f\left(\left[0, \frac{1}{2}\right]\right)$, $F_2(H) = f\left(\left[\frac{1}{2}, 1\right]\right)$, $F_3(K) = f(0)$ and $F_3(f([0,1])) = K$.

Let $\mathcal{S}_1 = (H, \{F_1, F_2, F_3\})$ where the functions F_1, F_2, F_3 are defined above. First we prove that H is the attractor of $\mathcal{S}_1$ and then that $\mathcal{S}_1$ is a topological iterated function system. We have that:

$$F_1(H) \cup F_2(H) \cup F_3(H) = f\left(\left[0, \tfrac{1}{2}\right]\right) \cup f\left(\left[\tfrac{1}{2}, 1\right]\right) \cup F_3(K) \cup F_3(f[0,1]) =$$
$$f([0,1]) \cup \{f(0)\} \cup K = f([0,1]) \cup K = H.$$

Thus H is the attractor of $\mathcal{S}_1$.

To prove that $\mathcal{S}_1$ is a topological iterated function system we only need to prove that $\bigcap\limits_{n\geq 1} F_{j_1 j_2 \ldots j_n}(H)$ is formed by one point for every $j_1, j_2, ..., j_n \in \{1, 2, 3\}$, where by $F_{j_1 j_2 \ldots j_n}$ we understand $F_{j_1 j_2 \ldots j_n} = F_{j_1} \circ F_{j_2} \circ \ldots \circ F_{j_n}$ because the first condition from the definition of a topological iterated function system is clearly satisfied by the compact set H.

We have the following properties concerning the compositions $F_{j_1 j_2 \ldots j_n} = F_{j_1} \circ F_{j_2} \circ \ldots \circ F_{j_n}$ with $j_1, j_2, ..., j_n \in \{1, 2, 3\}$:

a). $F_{33}(H) = F_3(F_3(H)) = F_3(F_3(K) \cup F_3(f([0,1]))) = F_3(f(0) \cup K) = F_3(K) = \{f(0)\}$.

b). For $j \in \{1, 2\}$:

$$F_{13j}(H) = F_1(F_3(F_j(H))) \subset F_1(F_3(f([0,1]))) \subset F_1(K) =$$
$$\{f(f_1(f^{-1}(p(K))))\} = \{f(f_1(f^{-1}(f(0))))\} = \{f(f_1(0))\} = \{f(0)\}.$$

Thus $F_{13j}(H) = \{f(0)\}$.

c). For $j \in \{1, 2\}$:

$$F_{23j}(H) = F_2(F_3(F_j(H))) \subset F_2(F_3(f([0,1]))) \subset F_2(K) =$$
$$\{f(f_2(f^{-1}(p(K))))\} = \{f(f_2(f^{-1}(f(0))))\} = \{f(f_2(0))\} = \left\{f\left(\tfrac{1}{2}\right)\right\}.$$

Hence $F_{23j}(H) = \{f(\frac{1}{2})\}$.

We will consider now the following two cases:

Case 1). $3 \in \{j_2, j_3, ..., j_n, ...\}$. Then let $j_n = 3$. We can have:

a). $3 \in \{j_{n-1}, j_{n+1}\}$.

Then if $j_{n-1} = 3$ we have that $F_{j_{n-1} j_n}(H) = F_{33}(H) = \{f(0)\}$ and if $j_{n+1} = 3$ we have that $F_{j_n j_{n+1}}(H) = F_{33}(H) = \{f(0)\}$. Thus $F_{j_{n-1} j_n j_{n+1}}(H)$ is formed by one point.

b). $3 \notin \{j_{n-1}, j_{n+1}\}$. Then

$F_{j_{n-1}j_nj_{n+1}}(H) =$
$= F_{j_{n-1}3j_{n+1}}(H) = \begin{cases} \{f(0)\}, \text{ if } j_{n-1} = 1 \\ \{f\left(\frac{1}{2}\right)\}, \text{ if } j_{n-1} = 2 \end{cases}$

Hence $F_{j_1j_2...j_{n+1}}(H)$ has one point and thus $\underset{n\geq 1}{\cap} F_{j_1j_2...j_n}(H)$ has one point.

Case 2). $3 \notin \{j_2, j_3, ..., j_n, ...\}$.

a). If $j_1 \neq 3$ then

$F_{j_1j_2...j_nj_{n+1}}(H) = (F_{j_1...j_n} \circ F_{j_{n+1}})(H) \subset F_{j_1...j_n}(f([0,1])) = (f \circ f_{j_1...j_n})([0,1]).$

Thus $\underset{n\geq 1}{\cap} F_{j_1j_2...j_n}(H) = \underset{n\geq 2}{\cap} F_{j_1j_2...j_n}(H) \subset \underset{n\geq 2}{\cap}(f \circ f_{j_1j_2...j_n})([0,1]) = f\left(\underset{n\geq 2}{\cap} f_{j_1j_2...j_n}([0,1])\right) = \{f(a)\}$, where $\{a\} = \underset{n\geq 2}{\cap} f_{j_1j_2...j_n}([0,1])$. We remark that the existence of a is assured by fact that $[0,1]$ is the attractor of $\mathcal{S} = \left([0,1]), \left\{f_1(x) = \frac{x}{2}, f_2(x) = \frac{x+1}{2}\right\}\right)$.

b). If $j_1 = 3$ then

$$\underset{n\geq 1}{\cap} F_{j_1j_2...j_n}(H) = \underset{n\geq 2}{\cap} F_{j_1j_2...j_n}(H) = F_3\left(\underset{n\geq 2}{\cap} F_{j_2j_3...j_n}(H)\right) = \{b\}$$

as it results from Case 2), point a).

Hence $\mathcal{S}_1 = (H, \{F_1, F_2, F_3\})$ is a topological iterated function system and H is its attractor.

We give now some examples.

Example 2.1. a). Consider Sierpinsky's triangle $T \subset \mathbb{R}^2$ and $H = T \cup [(a,b),(c,d)]$ where $T \cap [(a,b),(c,d)] \neq \varnothing$ and by $[(a,b),(c,d)]$ we understand the segment determined by the points (a,b) and (c,d). It is well known that T is the attractor of an iterated function system and we have that T is compact, connected and locally connected ([6]), thus T is a Peano space. Hence H is the attractor of a topological iterated function system formed by three functions. In particular, T is the attractor of a topological iterated function system formed by three functions, since T contains all his sides.

b). Consider the following set in the plane $\mathbb{R}^2$, $K = ([0,1] \times \{0\}) \cup \left(\underset{n\geq 1}{\cup} F_n\right)$, where $F_n = \left(\frac{1}{2^n}, 0\right) + F(a_n, k_n)$, $F(a_n, k_n) = \overset{k-1}{\underset{i=0}{\cup}} \left(\left\{\frac{ia}{k}\right\} \times [0,a]\right)$, $a_n = 2^{n^2-2^{2n}}$ and $k_n = 2^{2^{2n}}$ for every $n \geq 1$. In ([7]) it is proven that K cannot be the attractor of any iterated function system. But K is a compact, connected and locally connected set in the plane $\mathbb{R}^2$. Thus K is a Peano space. So $H = K \cup [(a,b),(c,d)]$ where $K \cap [(a,b),(c,d)] \neq \varnothing$ is the attractor of a topological iterated function system formed by three functions and, in particular, since the segment $[0,1] \times \{0\} \subset K$ we obtain that K is the attractor of a topological iterated function system formed by three functions.

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