DIFFERENT APPROACHES FOR A LINEAR MODEL FOR FINANCIAL PORTFOLIO OPTIMIZATION

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Abstract

This paper tries to present a linear model for financial portfolio in different type of approaches and to decide which approach is closer to reality. The first approach I propose is to consider that all variables are of continuous type. The second one considers an integer linear program. The third approach uses a special class of combinatorial optimization. For each type of approach I give a possible way to find an optimal solution. Finally, I give two examples of financial portfolio and the forms of optimization model for all proposed approaches.

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AMS classification: 90C05, 90C10, 91G10
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1. **Introduction**

In modern world one of the important activities is related to the financial portfolio. The financial portfolio is the way in which a company can invest a part of its profit on financial markets. Financial portfolio is also the way a person can uses his or her savings on stocks market. Also a financial portfolio is related to currencies.

An optimization for a financial portfolio can mean to determine a maximum benefits from portfolio or to minimize possible risks related to financial activities.

The simplest model for a financial portfolio is a linear model. In this case is very important to choose an approach close to reality and to have a good way to determine the optimal solution. The main problem is that the closer to reality the model is, the harder it is to solve the model. This problem can be eliminated by using a powerful computer with high performance software or by finding a less complex algorithm for a particular form of portfolio models.

This paper do not intent to give a solution for solving the portfolio linear models, but to present some different ways to see the problems of portfolio modeling.

2. **General financial portfolio problems**

In this paragraph I specify the considered context for financial portfolio problems.

The main portfolio starting date is represented by a given amount of financial resource. A characteristic of this resource is the currency used. To have a uniform treatment inside the problem I decided to use the same currency for all components of the studied portfolio.

Financial resources can be split in components and each part is used in different destination with specific effect or risk. The owner of financial resources can specify limitation about the same possible combination of portfolio components.

The financial problem goal is the maximization of the positive effect or minimization of the risk or negative effect.

Let me give now an example for a financial problem.

A company with financial activities has gained one resource in amount of 100,000 c,u. (currency unit) to be used on stocks market. Investment can be made in 6 type of stocks belong to three economic field, two type of stocks per economic field/ Economic policies make the following limitation as regard portfolio structure:

- Investment in first economic field must not exceed 70% of resource, which means 70,000 c.u.
- Part of resource allocated to second economic field must be less than 40,000 c.u.
- Minimal resource to be invest in third economic field represents at least 20%, which means 20,000 c.u.

Stocks of type 1 and 4 produce a level of 4% benefits/ Stocks of type 2 and 6 generate 5% profits/ Stock 5 generate 6% positive effect and stock 3 produce 8^% profits.

The company is interested in a portfolio structure with a specified condition and for which it generates maximum profits.

Let me give now the second example for a financial problem.

A person has savings representing 10,000 n.c.u (national currency units). He wishes to place his money in banking in separate accounts, each account for a given currency so that the risks to loose money to be minimum.

He decides to keep one part in national currency and the rest in currency like Euro, Pound, and Dollar. He also considers some limitation as regards his portfolio:

- Savings from national currency account and in extra European account must mot be over 6,000 n.c.u.
- Money from European foreign account must not exceed 5,000 n.c.u
- Savings in pounds and dollars accounts must be less than 3,000 n.c.u

Moneys may be lost as follows:

- 2% for national currency
- 4% for Euro
- 1,5% for Pound
- 2,3% for dollar.

3. **Financial portfolio model in linear programming**

A simple way to model a financial portfolio is to use linear programming. To do this I must make a formal description for portfolio problem. In general resource amount is designated by r and I consider that n the number of portfolio components.

Let x_1, x_2, …, x_n be model variables and for any i, $1 \le i \le n$, x_i represents the part from resource allocated to destination i. While all variables are allocated from existing resource mean that, for any i, $1 \le i \le n$, I have $x_i \ge 0$. Because resource is fully used I have now the first constraint:

$$x_1 + x_2 + ... + x_n = r \; .$$

Specified superior limits give a set of constraints with the following form:

$$Ax \le c$$

where A is a matrix with n columns.

Indicated inferior limits give a set of constraints

$$Bx \ge d$$

where B is a matrix with n columns.

The effect of allocation of a unit from resource to destination i is designated by e_i. So, the global effect for the portfolio is given by the linear function:

$$E = e_1 x_1 + e_2 x_2 + ... + e_n x_n$$

which must be optimized.

From the above consideration I obtain the following model:

$$\text{opt}\, E = e_1 x_1 + e_2 x_2 + ... + e_n x_n$$

$$\begin{cases} x_1 + x_2 + ... + x_n = r \\ Ax \le c \\ Bx \ge d \\ x_i \ge 0, i = 1, 2, ..., n \end{cases} \qquad (1)$$

It is obvious that this model is in linear programming for which I can use the simplex algorithm to find the optimal solution, if there is any.

If I consider the first example from section 1 than the resource for the portfolio is $r = 100,000$ which may be split in 6 type of stocks and so $n = 6$ and model variables are x_i, $1 \le i \le 6$. Implicit constraint to fully use the resource is

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100,000.$$

First and second limitations can be presented by constraints

$$x_1 + x_2 \le 70,000$$

and

$$x_3 + x_4 \le 40,000$$

so I can identify

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

and

$$c = \begin{pmatrix} 70,000 \\ 40,000 \end{pmatrix}.$$

The third limitation is written as

$$x_5 + x_6 \ge 20,000$$

and I have

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and

$$d = (20,000).$$

The effect of portfolio, which is a positive one, produces the following effect function

$$E = 0.04x_1 + 0.05x_2 + 0.08x_3 + 0.04x_4 + 0.06x_5 + 0.05x_6$$

and its optimization represent a maximization.

Now I can write linear model for first example from section 1 as

$$\max 0.04x_1 + 0,05x_2 + 0.08x_3 + 0.04x_4 + 0.06x_5 + 0.05x_6$$
$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100,000 \\ x_1 + x_2 \leq 70,000 \\ x_3 + x_4 \leq 40,000 \\ x_5 + x_6 \geq 20,000 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases} \tag{2}$$

For the second example in section 2, the resource is $r = 10,000$ and it may be placed in $n = 4$ different banking accounts. I suppose that all values involved in the problem are specified in national currency. Model variables are x_1 for national currency, x_2 for Euro, x_3 for Pound, x_4 for dollar and fully use of resource implicit constraint is

$$x_1 + x_2 + x_3 + x_4 = 10,000$$

.

Specified limitations for portfolio structure can be written as

$$x_1 + x_4 \geq 6,000$$
$$x_2 + x_3 \leq 5,000$$

and

$$x_3 + x_4 \leq 3,000.$$

So, I identify

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 5,000 \\ 3,000 \end{pmatrix}$$
$$d = (6,000).$$

The effect function is

$$E = 0.02x_1 + 0.04x_2 + 0.015x_3 + 0.023x_4$$

and optimization is minimization because the effects are negative.

I obtain the following model fort portfolio specified in second example in section 2:

$$\min E = 0.02x_1 + 0.04x_2 + 0.015x_3 + 0.023x_4$$
$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 10,000 \\ x_2 + x_4 \le 5,000 \\ x_3 + x_4 \le 3,000 \\ x_1 + x_4 \ge 6,000 \\ x_1, x_2, x_3, x_4 \ge 0 \end{cases} \qquad (3)$$

4. **Financial portfolio model in (pure) integer linear programming**

In financial practice all operation are specified in currency. Even there is any currency with exception; most currencies are indicated as number with exactly two decimals. This it is possible because currency has a subunit (the subunit for dollar is the cent and 1 dollar has 100 cents, the subunit for the Romanian currency, *leu*, is *ban* and 1 *leu* has 100 *bani*, and so on). This fact allows me to assume that values measured by currency may be considered as integer value. If it is not so, I can use the smallest subunit for considered currency and convert currency values in their subunits and these values are surely integers.

The above considerations tell me that in linear model proposed for financial portfolio I have no justification to consider that model variables are continuous on. More than that, because all variables are non negative and integer I can say that all variables are natural numbers.

Changing variables domain make resulted model closer to reality from financial portfolio. With this changing general model (1) became:

$$\text{opt } E = e_1x_1 + e_2x_2 + \cdots + e_nx_n$$
$$\begin{cases} x_1 + x_2 + \cdots + x_n = r \\ Ax \le c \\ Bx \ge d \\ x_i \in \mathrm{Z}, \mathrm{i} = 1, 2, ..., \mathrm{n} \end{cases} \qquad (4)$$

To solve model (4) I can apply any procedure for integer linear programming which is more complex then the simplex algorithm.

If I consider the model (4) for the financial portfolio then model (1) represent a relaxation of the initial problem.

Due to optimization theory results, if x^* is the optimal solution for relaxation model (1) and $\overline{x}$ is the optimal solution for integer model (4) then, for every i, $1 \le i \le n$, if $x_i^* \in Z$ then $\overline{x}_i = x_i^*$. So a simple procedure to solve model (4) is to repeat the following two steps until all variables are fixed to integer, starting with model (1) without fixed variables.

1. I isolate part of model (1) which do not contain any fixed variable and solve this part using simplex algorithm.
2. I fix in model (1) all integer component from solution obtained in step 1. Choose one of non integer component from solution obtained in step 1. and fix it to its integer part (in this choice it is better to take the component with the smallest decimal part.

If I consider the first example in section 2 with linear model (2) then integer linear model is:

$$\begin{aligned} &\max 0.04x_1 + 0.05x_2 + 0.08x_3 + 0.04x_4 + 0.06x_5 + 0.05x_6 \\ &\sum_{i=1}^{6} x_i = 100{,}000 \\ &x_1 + x_2 \le 70{,}000 \\ &x_3 + x_4 \le 40{,}000 \\ &x_5 + x_6 \ge 20{,}000 \\ &x_i \in Z, \mathrm{i} = 1,2,...,6 \end{aligned} \qquad (5)$$

For the second example in section 2 integer linear model is obtained from (3) and it has the following form:

$$\begin{aligned}
&\min E = 0.02x_1 + 0.04x_2 + 0.015x_3 + 0.023x_4 \\
&\sum_{i=1}^{4} x_i = 10{,}000 \\
&x_2 + x_4 \le 5{,}000 \\
&x_3 + x_4 \le 3{,}000 \\
&x_1 + x_4 \ge 6{,}000 \\
&x_i \in Z, \mathrm{i} \in 1..4.
\end{aligned} \tag{6}$$

5. **A new approach for financial portfolio model**

The general case of integer variable for the financial portfolio represents a too large domain because the portfolio resource specified in the portfolio problem is constant. More than that, even for a variable amount for financial resources, it can not be considered as infinite one. This is why I can consider that there exists a finite limit for every component of the portfolio.

In extension I can consider that all but one component is zero and so no zero component value is equal with r. Such case allows me to replace domains condition $x_i \in Z$ with $x_i \in \{0,1,...,r\}$ for every $i = 1,2,...,n$. Because obtained model (from (1) or (4))

$$\operatorname{opt} E = e_1x_1 + e_2x_2 + ... + e_nx_n$$
$$\begin{cases} x_1 + x_2 + ... + x_n = r \\ Ax \le c \\ Bx \ge d \\ x_i \in \{0,1,...,r\}, i = 1,2,...,n \end{cases} \tag{7}$$

has all variables integer and in bounded domains, model (7) belongs to combinatorial optimization. It could be seen that general models (1) and (4) became relaxations for model (7) and so, the last one is closer to the real situation of the financial portfolio.

I must observe that model (7) contains a special constraint

$$x_1 + x_2 + ... + x_n = r$$

for which, in some papers S. Bârză use the name *constant sum constraint*.

Also, I observe that all variables have the same domain and upper bound of variables domain is equal with right hand constant sum constraint. S. Bârză [2], includes all combinatorial model with these characteristics in a special class named *percentage programming*.

For this moment I do not find specific algorithms for percentage programming, but I think that for solving the model (7) it may be useful any procedure for combinatorial optimization.

As regards the examples in section 2, the first one produces the following model in percentage programming:

$$\max 0.04x_1 + 0,05x_2 + 0.08x_3 + 0.04x_4 + 0.06x_5 + 0.05x_6$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100,000 \\ x_1 + x_2 \le 70,000 \\ x_3 + x_4 \le 40,000 \\ x_5 + x_6 \ge 20,000 \\ x_1, x_2, x_3, x_4, x_5, x_6 \in \{0,1,...,100000\} \end{cases} \tag{8}$$

For the second example I obtain the model

$$\min E = 0.02x_1 + 0.04x_2 + 0.015x_3 + 0.023x_4$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 10,000 \\ x_2 + x_4 \le 5,000 \\ x_3 + x_4 \le 3,000 \\ x_1 + x_4 \ge 6,000 \\ x_1, x_2, x_3, x_4 \in \{0,1,...,10000\} \end{cases} \tag{9}$$

6. **Summary**

The goal of this paper was to present alternative approaches for financial portfolio and it includes three different classes in optimizations which can be included one in other. So percentage programming may be included in integer linear programming which is included in (continuous) linear programming. This inclusion can be seen in relaxation of variables domains. In my case variables domains inclusion is

$$\{0,1,2,...,r\} \subset Z \subset R_+ .$$

On the other hand, I can give two ways to say which model is better.

1. one model is better if it is closer to reality – from this point of view I can say that (7) (percentage programming model) is better than (4) (integer linear model) and (1) (continuous linear model);

2. one model is better if it is easier to solve – in this case I can say that (1) (continuous linear model) is better than (4) (integer linear model) and (7) (percentage programming model).

This paper does not wish to give procedures for model solving. It only suggests shortly the way in which models can be solved for the need to appreciate the quality of better model from this point of view. I included two examples only to illustrate my approaches.

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